15 Questions

AoPM

1) Consider sum of i.i.d. Bernoulli random variables s.t. $\forall i \in \{1,2,\dots,n\}$, $X_i \sim \mathrm{Bern}(p)$ and $X \triangleq \sum_{i=1}^{n} X_i$. You are given the bound

ⓐ $\Pr[X \geq np + \lambda] \overset{(\lambda>0)}{\leq} e^{-nD(p+\frac{\lambda}{n} \| p)}$.

By using ⓐ, prove that

ⓑ $\Pr[X \leq np - \lambda] \leq e^{-nD(1-p+\frac{\lambda}{n} \| 1-p)}$. [Lecture Notes pages 18-19]

2) Consider sum of Bernoulli independent r.v.s s.t. $\forall i \in \{1,2,\dots,n\}$, $X_i \sim \mathrm{Bern}(p_i)$ and $X \triangleq \sum_{i=1}^{n} X_i$. Define $\mathbb{E}[X] = \sum_{i=1}^{n} p_i \triangleq n\bar{p}$. Prove that we can use the bound given in 1).ⓐ above by replacing p with $\bar{p}$. [Lecture Notes pages 22-23]

3) Consider the "Interference model" example from the lecture notes pages 25-27, where

$$\underbrace{y(t)}_{\text{received}} = \underbrace{a(t)}_{\text{sent}} + \overbrace{\sum_{i=1}^{n} h_i \cdot b_i(t)}^{\text{interference}}$$

- $a, b_1, b_2, \dots, b_n \in \{\mp 1\}$ and uniformly distributed,
- $h_i \leq 1$ and are known at RX.
- RX wants to decide whether $a(t) = +1$ or -1.
- RX uses the detector $y(t) \overset{?}{\gtrless} 0$

$\Downarrow$

$\Pr[\text{error in the decision}] \leq ?$

4) Define $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ as two independent r.v.s with $\lambda < \mu$. Which random variable do you expect to be larger? Prove that

$$\Pr[X \geq Y] \leq e^{-(\sqrt{\mu}-\sqrt{\lambda})^2}.$$

[In a self-study question]

5) A fixed point of a permutation $\pi : [1:n] \to [1:n]$ is a value for which $\pi(x) = x$. Find the variance of the number of fixed points of a permutation that is chosen uniformly at random from all permutations. [In a tutorial]

6) Given $\mu \in \mathbb{R}$ and $\sigma, \lambda > 0$, construct a r.v. X with mean μ and variance σ^2 s.t. the Cantelli's inequality is satisfied with equality.

7) The cumulative distribution function (CDF) of a r.v. is defined as $F_X(x) = \Pr[X \leq x]$. For any non-negative r.v. X, prove that

$$\mathbb{E}[X] = \int_0^\infty \left(1 - F_X(x)\right) dx.$$

[A self-study question]

8) Let X be a standard normal r.v. Find the moment generating function of the r.v. X^2, i.e., calculate $\mathbb{E}[e^{tX^2}]$. For which values of t does the moment generating function exist?

[A self-study question]

9) Recall the "Rich gets richer" example from the lecture notes pages 60-62, where in a basket there are W_k white balls and B_k black balls at time $k \in \{1, 2, \dots, n\}$. Pick a ball. If the ball is black, put the black ball back and add a new black ball. Similarly, if the ball is white, put the white ball back and add a new white ball. Prove that $Z_k = \dfrac{W_k}{W_k + B_k}$ is a martingale. Is $\tilde{Z}_k = \dfrac{B_k}{W_k + B_k}$ also a martingale?

10) Consider the "Balls and Bins" example from Lecture Notes pages 78-79, where m balls are thrown into n bins independently and uniformly at random. Define $F \triangleq \left\{ \begin{array}{l} \text{the number of empty bins after all balls} \\ \text{are thrown} \end{array} \right\}$.

$$\Downarrow$$

$$\Pr\left[\,|F - \mathbb{E}[F]| \geq \lambda\,\right] \leq ?$$

11) Consider a sample space with the following six points: $(-1,0), (0,0), (0,1), (1,0), (1,1), (2,1)$. We choose a point (X,Y) at random from this set with probabilities

$$\Pr_{(X,Y)}(-1,0) = \frac{2}{7},$$

$$\Pr_{(X,Y)}(0,0) = \Pr_{(X,Y)}(0,1) = \Pr_{(X,Y)}(1,0) = \Pr_{(X,Y)}(1,1) = \Pr_{(X,Y)}(2,1) = \frac{1}{7}.$$

a) Prove ~~that~~ if X and Y are independent.

b) Compute and compare $\mathbb{E}\left[\mathbb{E}[Y|X]^2\right]$ and $\mathbb{E}\left[\mathbb{E}[Y|X^2]^2\right]$.

Can you argue, without any calculations, which one should be larger?

12) Given an undirected graph G with n vertices and e edges, there exists a cut with value at least $e/2$. Prove it.

[Lecture Notes pages 86-87]

13) If the number of vertices in a graph N $\wedge$ can be written as $N = n - \binom{n}{3} 2^{-\binom{3}{2}+1}$ for some $n \geqslant 3$, then there exists a coloring of N vertices with no mono-chromatic triangle (Assume that there are only 2 possible colors). Prove it.

[Lecture notes page 95]

14) Let $X \triangleq \sum_{i=1}^{n} X_i$, where $X_i \in \{0,1\}$ that can be dependent.

Prove that

$$Pr[X \geqslant 1] \geqslant \sum_{i=1}^{n} \frac{Pr[X_i = 1]}{\mathbb{E}[X \mid X_i = 1]}.$$

[Lecture notes pages 105-106]

15) $X_1, X_2, \ldots, X_n$ are independent r.v.s uniform over $[-1, 1]$. Define

$$Y = \sum_{i \neq j \in \{1,2,\ldots,n\} : |i-j| \leq 3} X_i \cdot X_j$$

(a) Find $\mathbb{E}[Y]$.

(b) Find an upper bound on $\Pr[Y > nc]$ for some given $c > 0$. The upper bound should vanish exponentially fast as n tends to infinity.