Musterlösung Aprilklausur
Rechenteil ITPDGL

(R 1)

1. Aufgabe

Gegeben ist das AWP $y''+2y'+y=4e^x$, $y(0)=1$, $y'(0)=0$.

Anwendung der Laplace-Transformation:

$$\mathcal{L}[y''+2y'+y](s)=\mathcal{L}[4e^x](s).$$

Setze $\mathcal{L}[y](s)=\underline{Y}(s)$, Linearität und Ableitungssatz führt zu:

$$s^2\underline{Y}(s)-s\,y(0)-y'(0)+2\big(s\underline{Y}(s)-y(0)\big)+\underline{Y}(s)=\frac{4}{s-1}.$$

$$\Rightarrow\quad \underline{Y}(s)\left(s^2+2s+1\right)-s-2=\frac{4}{s-1}$$

$$\Rightarrow\quad \underline{Y}(s)=\frac{\frac{4}{s-1}+s+2}{s^2+2s+1}=\frac{4+(s-1)(s+2)}{(s^2+2s+1)(s-1)}=$$

$$\frac{4+s^2+2s-s-2}{(s+1)^2(s-1)}=\frac{s^2+s+2}{(s+1)^2(s-1)}\overset{\boxed{\text{PBZ}}}{=}\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{(s+1)^2}$$

$A,B,C\in\mathbb{R}$

$$s^2+s+2=A(s+1)^2+B(s+1)(s-1)+C(s-1)$$

$A=1$, $B=0$, $C=-1$

$$\underline{Y}(s)=\frac{1}{s-1}-\frac{1}{(s+1)^2}\quad\Rightarrow\quad \underline{\underline{y(x)=e^x-xe^{-x}}}$$

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2 Aufgabe

a) $x(n) = \begin{cases} 2^n & n \text{ gerade} \\ \frac{1}{3^n} & n \text{ ungerade} \end{cases}, \quad n \in \mathbb{N}_0$

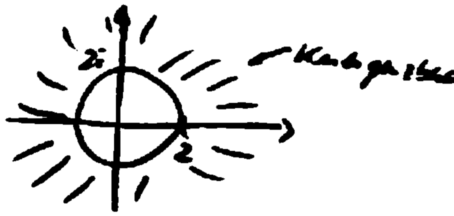
$$\mathcal{Z}[x(n)](z) = \sum_{n=0}^{\infty} x(n) \frac{1}{z^n} = \sum_{l=0}^{\infty} x(2l) \frac{1}{z^{2l}} + \sum_{l=0}^{\infty} x(2l+1) \frac{1}{z^{2l+1}}$$

$$= \sum_{l=0}^{\infty} 2^{2l} \frac{1}{z^{2l}} + \sum_{l=0}^{\infty} \frac{1}{3^{2l+1}} \frac{1}{z^{2l+1}} = \sum_{l=0}^{\infty} \left(\frac{2}{z}\right)^{2l} + \sum_{l=0}^{\infty} \frac{1}{(3z)^{2l+1}}$$

$$= \frac{1}{1-\left(\frac{2}{z}\right)^2} + \frac{1}{3z} \frac{1}{1-\left(\frac{1}{3z}\right)^2}, \quad \left|\frac{2}{z}\right|^2 < 1, \quad \left|\frac{1}{3z}\right|^2 < 1$$

$$\boxed{|z| > 2}, \quad |z| > \frac{1}{3}$$

$$= \frac{z^2}{z^2-4} + \frac{3z}{9z^2-1}, \quad |z|>2$$



3 Aufgabe

$$\int_{-\infty}^{+\infty} y(\tau)\, y(t-\tau)\, d\tau = e^{-2t^2}$$

Anwendung der Fouriertransformation liefert:

$$F\left[\int_{-\infty}^{+\infty} y(\tau)\, y(t-\tau)\, d\tau\right](\omega) = F\left[e^{-2t^2}\right](\omega).$$

Die Faltungsregel angewendet ergibt

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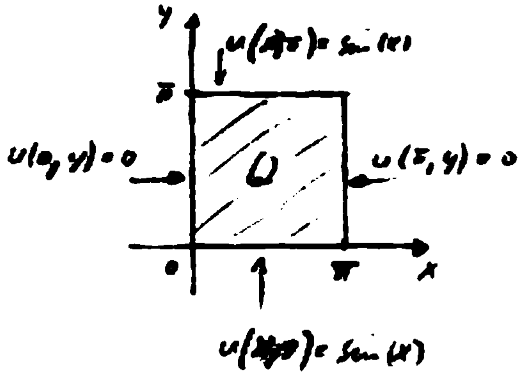
$$F(\omega)\cdot F(\omega) = \sqrt{\frac{\pi}{2}}\, e^{-\frac{\omega^2}{8}} \qquad \text{mit } F(\omega) = \mathcal{F}[y(t)](\omega)$$

Also $$F(\omega) = \pm\left(\frac{\pi}{2}\right)^{\frac{1}{4}}\cdot e^{-\frac{\omega^2}{16}} = \pm\frac{\left(\frac{\pi}{2}\right)^{\frac{1}{4}}\sqrt{\frac{\pi}{4}}}{\sqrt{\frac{\pi}{4}}}\, e^{-\frac{\omega^2}{4\cdot 4}},$$

Mit dem Hinweis und $a=4$ folgt:

$$y(t) = \pm\frac{\left(\frac{\pi}{2}\right)^{\frac{1}{4}}}{\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}\cdot e^{-4t^2} = \pm\frac{2^{\frac{3}{4}}}{(\pi)^{\frac{1}{4}}}\, e^{-4t^2}$$

$$u_{xx} + u_{yy} = 0$$



Produktansatz $u(x,y) = X(x)\cdot Y(y)$ auf die Potentialgleichung anwenden:

$$X'' Y + X Y'' = 0 \iff \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \in \mathbb{R}$$

Also $\frac{X''}{X} = \lambda \Rightarrow X'' - \lambda X = 0$, $X(0) = X(\pi) = 0$

$\Rightarrow \lambda < 0$

Setze $\lambda = -c^2$, $c \in \mathbb{R}$

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4) $X'' + c^2 X = 0$

$X(x) = a\cos(cx) + b\sin(cx) \quad , \quad a,b \in \mathbb{R}$

$X(0) = 0 \Rightarrow a = 0 \Rightarrow X(x) = b\sin(cx)$

$X(\pi) = 0 \Rightarrow \sin(c\pi) = 0 \Rightarrow c = k \in \mathbb{Z}$

$b=0 \Rightarrow X \equiv 0$ (triviale Lsg)

$\Rightarrow X_k(x) = b_k \sin(kx) \quad , \quad k \in \mathbb{Z}, b_k \in \mathbb{R}.$

Löse $\frac{Y''}{Y} = \lambda = -c^2 = -k^2$ also $\frac{Y''}{Y} = k^2$.

$Y_k'' = k^2 Y_k \Rightarrow Y_k(y) = A_k e^{ky} + B_k e^{-ky} \quad , \quad A_k B_k \in \mathbb{R}, k \in \mathbb{Z}$

$\Rightarrow u_k(x,y) = \sin(kx)\left(A_k e^{ky} + B_k e^{-ky}\right)$ oBdA $b_k = 1$.

A_k und B_k mit den verbleibenden Randbedingungen bestimmen

Superpositionsprinzip $\quad u(x,y) = \sum_{k=1}^{\infty} \sin(kx)\left(A_k e^{ky} + B_k e^{-ky}\right)$

$\sin(x) = u(x,0) = \sum_{k=1}^{\infty} \sin(kx)\,(A_k + B_k) \Rightarrow A_k + B_k = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases}$ I

$\sin(x) = u(x,\pi) = \sum_{k=1}^{\infty} \sin(kx)\left(A_k e^{k\pi} + B_k e^{-k\pi}\right) \Rightarrow$

$A_k e^{k\pi} + B_k e^{-k\pi} = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases}$ II

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4) $I) \Rightarrow A_1 = 1 - B_1$

$II) \Rightarrow A_1 e^{\pi} + B_1 e^{-\pi} = 1$

$\Leftrightarrow (1-B_1) e^{\pi} + B_1 e^{-\pi} = 1$

$\Leftrightarrow -B_1 (e^{\pi} - e^{-\pi}) = 1 - e^{\pi}$

$\Leftrightarrow B_1 = \dfrac{e^{\pi}-1}{e^{\pi}-e^{-\pi}} \qquad A_1 = 1 - \dfrac{e^{\pi}-1}{e^{\pi}-e^{-\pi}} = \dfrac{1-e^{-\pi}}{e^{\pi}-e^{-\pi}}$

$I) \Rightarrow A_k = -B_k, \; k \neq 1,$

$II) \Rightarrow A_k e^{k\pi} + B_k e^{-k\pi} = 0 \quad k \neq 1$

$\Rightarrow A_k (e^{k\pi} - e^{-k\pi}) = 0 \Rightarrow A_k = B_k = 0 \quad k \neq 1$

$\Rightarrow u(x,y) = \sin(x) \left(\dfrac{1-e^{-\pi}}{e^{\pi}-e^{-\pi}} e^{y} + \dfrac{e^{\pi}-1}{e^{\pi}-e^{-\pi}} e^{-y} \right)$

$\Leftrightarrow u(x,y) = \dfrac{\sin(x)}{2\sinh(\pi)} \left(e^{y} - e^{y-\pi} + e^{-(y-\pi)} - e^{-y} \right)$

$\Leftrightarrow u(x,y) = \dfrac{\sin(x)}{2\sinh(\pi)} \cdot \left(2\sinh(y) - 2\sinh(y-\pi) \right)$

$\Leftrightarrow u(x,y) = \dfrac{\sin(x) \left(\sinh(y) - \sinh(y-\pi) \right)}{\sinh(\pi)}$

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