Musterlösung Rechenteil ITPDG

WS 03/04 Februar-Klausur

(R1)

Aufgabe 1 $\quad t\,y'(t) - 3y(t) = 0 \quad , \quad y(1) = 2$

Anwenden der Laplacetransformation auf die DGL:

$$\mathcal{L}[t\,y'(t)](s) = -\frac{d}{ds}\mathcal{L}[y'(t)](s)$$

$$= -\frac{d}{ds}\Big(s\,\underbrace{\mathcal{L}[y(t)](s)}_{=:L(s)} - y(0)\Big) = -\frac{d}{ds}\Big(s\,L(s)\Big)$$

$$\Rightarrow \mathcal{L}[t\,y'(t)](s) = -L(s) - s\,L'(s)$$

Insgesamt ergibt sich:

$$\mathcal{L}[t\,y'(t) - 3y(t)](s) = \mathcal{L}[0](s)$$

$$\Rightarrow -L(s) - s\,L'(s) - 3L(s) = 0$$

$$\Rightarrow L'(s) = -\frac{4}{s}L(s) \quad \Rightarrow \quad \frac{L'(s)}{L(s)} = -\frac{4}{s} \quad (L(s) \neq 0)$$

(trennbare DGL)

$$\Rightarrow \ln|L(s)| = -4\ln|s| + c \quad , \quad c \in \mathbb{R}$$

$$\Rightarrow L(s) = \frac{\tilde{c}}{s^4} \quad , \quad \tilde{c} = \pm e^c \quad , \quad \tilde{c} \in \mathbb{R}\setminus\{0\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{\tilde{c}}{s^4}\right](t) = \tilde{c}\,\frac{t^3}{3!} = \tilde{c}\,\frac{t^3}{6} \quad , \quad \tilde{c} \in \mathbb{R}$$

$\tilde{c} = 0$ führt zur offensichtlichen Lösung $y \equiv 0$ ✓

$y(1) = 2 \Rightarrow \tilde{c} = 12 \Rightarrow y(t) = 2t^3$ löst das AWP.

Aufgabe 2 $\quad y_{n+2} + 2y_{n+1} + y_n = 1, \quad y_0 = 1, \quad y_1 = -1$

Anwenden der z-Transformation

$$\left.\begin{aligned}
&\mathcal{Z}[y_n](z) = F^*(z) \\
&\mathcal{Z}[y_{n+1}](z) = z\left(F^*(z) - y_0\right) \\
&\mathcal{Z}[y_{n+2}](z) = z^2\left(F^*(z) - y_0 - \frac{y_1}{z}\right)
\end{aligned}\right\} \text{Verschiebungssatz}$$

$$\Rightarrow z^2 F^*(z) - z^2 y_0 - z y_1 + 2zF^*(z) - 2z y_0 + F^*(z) = \mathcal{Z}[1](z).$$

$$\Rightarrow F^*(z)\left(z^2 + 2z + 1\right) \quad z^2 + z - 2z = \frac{1}{1 - \frac{1}{z}}$$

$$\Rightarrow F^*(z)\,(z+1)^2 - z^2 - z = \frac{z}{z-1}$$

$$\Rightarrow F^*(z) = \frac{1}{(z+1)^2}\left(\frac{z}{z-1} + z(z+1)\right)$$

$$\Rightarrow F^*(z) = \frac{z}{(z-1)(z+1)^2} + \frac{z}{(z+1)} \qquad \text{(Partialbruchzerlegung)}$$

$$\Rightarrow F^*(z) = \frac{\frac{1}{4}}{z-1} + \frac{-\frac{1}{4}}{z+1} + \frac{\frac{1}{2}}{(z+1)^2} + \frac{z}{z+1}, \quad \text{Subst. } z = \frac{1}{w}$$

$$\Rightarrow F^*\left(\frac{1}{w}\right) = \frac{1}{4}\,\frac{w}{1-w} - \frac{1}{4}\,\frac{w}{1+w} + \frac{1}{2}\,\frac{w^2}{(1+w)^2} + \frac{1}{1+w}$$

$$\Rightarrow F^*\left(\frac{1}{w}\right) = \frac{w}{4}\,\frac{1}{1-w} - \frac{w}{4}\,\frac{1}{1-(-w)} - \frac{w^2}{2}\left(\frac{1}{1+w}\right)' + \frac{1}{1-(-w)}$$

$$\Rightarrow F^*\left(\frac{1}{w}\right) \underset{|w|<1}{=} \frac{w}{4}\sum_{n=0}^{\infty} w^n - \frac{w}{4}\sum_{n=0}^{\infty}(-w)^n - \frac{w^2}{2}\left(\sum_{n=0}^{\infty}(-w)^n\right)' + \sum_{n=0}^{\infty}(-w)^n$$

$$\Rightarrow F^*\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty} \frac{1}{4} w^{n+1} + \frac{(-1)^{n+1}}{4} w^{n+1} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\cdot n}{2} w^{n+1} + \sum_{n=0}^{\infty} (-1)^n w^n$$

(R3)

$$\Rightarrow F^*\left(\frac{1}{w}\right) = \sum_{n=1}^{\infty} \frac{1}{4} w^n + \frac{(-1)^n}{4} w^n + \frac{(-1)^n (n-1)}{2} w^n +$$

$$\sum_{n=1}^{\infty} (-1)^n w^n + 1$$

$$\Rightarrow F^*\left(\frac{1}{w}\right) = \left[\sum_{n=1}^{\infty} \left(\frac{1}{4} + \frac{(-1)^n}{4} + \frac{(-1)^n (n-1)}{2} + (-1)^n\right) w^n\right] + 1$$

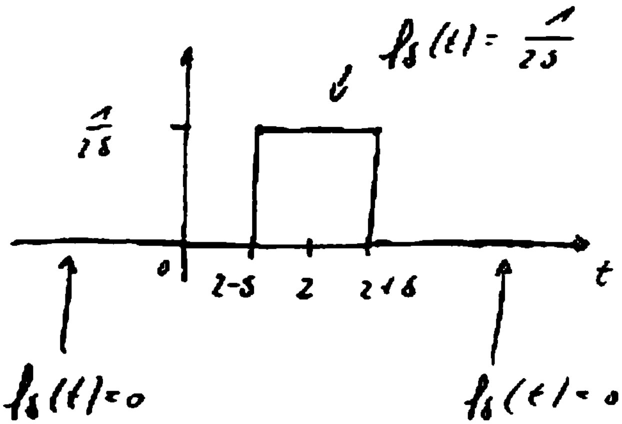
$$\Rightarrow \quad y_0 = 1$$

$$y_n = \frac{1}{4} + \frac{3}{4}(-1)^n + \frac{(-1)^n n}{2}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad \underline{\underline{y_n = \frac{1}{4} + (-1)^n \left(\frac{3}{4} + \frac{n}{2}\right)}} \qquad n = 0, 1, 2, \ldots$$

Aufg. 3

(R4)



$$F[f_\delta(t)](\omega)=\int_{-\infty}^{+\infty} f_\delta(t)\, e^{-i\omega t}\, dt$$

$$=\int_{2-\delta}^{2+\delta} \frac{1}{2\delta}\, e^{-i\omega t}\, dt = \frac{1}{2\delta}\cdot\frac{-1}{i\omega}\, e^{-i\omega t}\Big|_{2-\delta}^{2+\delta}$$

$$=\frac{-1}{2\delta i\omega}\left(e^{-i\omega(2+\delta)}-e^{-i\omega(2-\delta)}\right)$$

$$\lim_{\delta\to 0} F[f_\delta(t)](\omega)=\lim_{\delta\to 0}\frac{e^{-i\omega(2-\delta)}-e^{-i\omega(2+\delta)}}{2\delta i\omega}$$

$$=\lim_{\delta\to 0}\frac{i\omega\cdot e^{-i\omega(2-\delta)}+i\omega\cdot e^{-i\omega(2+\delta)}}{2i\omega}=e^{-2i\omega}$$

"$\frac{0}{0}$" Regel von l'Hospital

Aufg. 4a) $u_t = 4u_{xx} + 2tx$

$u(0,t)=0, \quad u(\pi,t) = \pi t^2$

$u(x,0) = 3\sin(5x)$

Setze $v(x,t) = u(x,t) - \hat{u}(x,t)$ mit

$\hat{u}(x,t) = a(t)\,x + b(t)$, $a(t), b(t)$ so bestimmen, dass

$v(0,t) = v(\pi,t) = 0$ gilt.

$v(0,t) = u(0,t) - \hat{u}(0,t) = 0 - b(t) \overset{!}{=} 0 \Rightarrow b(t) = 0$

$v(\pi,t) = u(\pi,t) - \hat{u}(\pi,t) = \pi t^2 - a(t)\pi \overset{!}{=} 0$

$\Rightarrow a(t) = t^2 \Rightarrow \hat{u}(x,t) = t^2 x$

$\Rightarrow \boxed{v(x,t) = u(x,t) - t^2 x} \quad \Rightarrow \boxed{v_{xx} = u_{xx}}$

DGL für v bestimmen:

$$
\begin{aligned}
v_t(x,t) &= u_t(x,t) - 2tx \\
&= 4u_{xx}(x,t) + 2tx - 2tx \\
&= 4u_{xx}(x,t) \\
&= 4v_{xx}(x,t)
\end{aligned}
$$

$v(0,t) = v(\pi,t) = 0$

$v(x,0) = u(x,0) = 3\sin(5x)$

Aufg. 4b)

$v_t = 4 v_{xx}$

$v(0,t) = v(\pi, t) = 0$

$v(x,0) = 3 \sin(5x)$

(R6)

Produktansatz: $v(x,t) = X(x)\, T(t)$

$\Rightarrow X T' = 4 X'' T \Rightarrow \frac{T'}{4T} = \frac{X''}{X} = \lambda \in \mathbb{R}$

Ortsgleichung lösen $X'' - \lambda X = 0$ $\quad \underbrace{X(0) = X(\pi) = 0}_{\Rightarrow \lambda < 0}$

Setze $\lambda = -c^2$

$X'' + c^2 X = 0$

$\Rightarrow X(x) = a \cos(cx) + b \sin(cx)$

$X(0) = 0 \Rightarrow a = 0 \quad X(\pi) = 0 \Rightarrow b \sin(c\pi) = 0$

$b = 0 \Rightarrow X \equiv 0 \Rightarrow v \equiv 0$ (triviale Lsg.) Sei $b \neq 0$

$\Rightarrow \sin(c\pi) = 0 \Rightarrow c_k = k \in \mathbb{Z} \Rightarrow \boxed{\lambda_k = -k^2}$

$\boxed{X_k(x) = b_k \sin(kx)}$

Zeitgleichung lösen $T' = 4\lambda T$

$T_k' = -4k^2 T_k \Rightarrow \boxed{T_k(t) = A_k e^{-4k^2 t}}$

$\Rightarrow v_k(x,t) = \underbrace{b_k}_{=1 \text{ oBdA.}} \sin(kx)\, A_k e^{-4k^2 t}$

Superpositionsprinzip

$$\Rightarrow v(x,t) = \sum_{k=1}^{\infty} A_k \sin(kx)\, e^{-4k^2 t}$$

$$v(x,0) = \sum_{k=1}^{\infty} A_k \sin(kx) \overset{!}{=} 3\sin(5x)$$

$$\Rightarrow A_k = 0 \quad k \neq 5 \quad , \quad A_5 = 3$$

$$\Rightarrow \boxed{v(x,t) = 3\sin(5x)\, e^{-100t}}$$

<u>Aufg 4 c)</u>

Das Rand-Anfangswertproblem mit homogenen Randwerten aus 4a) ist genau das Rand-Anfangswertproblem aus 4b). Aus der Lösung von 4b) kann somit die Lösung von 4a) gewonnen werden:

$$u(x,t) = v(x,t) + \tilde{u}(x,t)$$

$$\leadsto \boxed{u(x,t) = 3\sin(5x)\, e^{-100t} + t^2 x}$$