Prüfungsfragebogen zu

Prüfungsfach (bitte leserlich ;))
Modern Signal Processing for Communications (MSPC)

- [x] mündlich
- [ ] schriftlich
- [ ] Nachprüfung

Datum: 02/2021
Prüfer: cavalcante
Prüfungsdauer: 30 min
Studiengang: ETEC M.Sc.

Vorbereitung

a) Regelmäßiger Besuch der Lehrveranstaltung? [x] Ja [ ] Nein

b) Auswirkungen von a): [x] Positiv [ ] Keine [ ] Negativ

c) Dauer der Vorbereitung: 1.5 weeks [x] Alleine [ ] In der Gruppe

d) Vorkenntnisse aus anderen Fächern/Praxiserfahrung?

basic signal processing

e) Welche Hilfsmittel wurden benutzt? (*Literatur, Internetseiten etc.*)

/

f) Welche Tipps würdest du zur Vorbereitung geben?

know the meaning and interconnections of the exercises

Prüfung

a) Gab es Absprachen über Form oder Inhalt und wurden sie eingehalten?

only excersizes will be asked. → mostly true

b) Ratschläge zum Verhalten während der Prüfung:

c) Prüfungsstil: (*Atmosphäre, klare oder unklare Fragestellungen, Detailwissen oder Zusammenhänge, gezielte Zwischenfragen, Hilfestellung, gezielte Fragen bei Wissenslücken, …?*)

- he will give hints if you are not perfectly sure
- only basic way of solving needed.
  → $P_r(x)$ is already an answer. don't go too deep!

Verschiedenes

a) Welche Note hast du bekommen? (*natürlich optional*) 1.0

b) Empfandest du die Bewertung als angemessen? [x] Ja [ ] Nein (*warum nicht?*)

c) Kannst du die Prüfung weiterempfehlen? [x] Ja (*wem besonders?*) [ ] Nein (*warum nicht?*)

→ really cool signal processing.

d) Hast du darüber hinaus Tipps und Bemerkungen auf Lager?

Inhalt der Prüfung: Bitte gib möglichst viele Fragen an. Wo wurden Herleitungen verlangt, und wo wurde nach Beweisen gefragt? (*Wenn der Platz nicht reicht kannst du auch gerne weitere Blätter verwenden. Am besten zusammengeheftet und durchnummeriert.*)

(*) What is more general? (normed vector space / Banach / pre-Hilbert)

→ exercise 1)

→ properties of a metric

→ properties of a norm

(*)

→ properties of an inner product.

can you find an inner product for every norm (e.g. $\|\cdot\|_\infty$)? no.

→ ex ~~18~~

→ ex 1~~8~~9

→ ex 22

→ ex 23

} use this

→ ex 25

} and this

} to further proof

Vielen Dank für deine Bemühungen! **Deine Kommilitoninnen und Kommilitonen.**

Exercises - Summer 2020

(1) Relate metric spaces, Banach spaces, pre-Hilbert spaces, and Hilbert spaces.

(2) In a metric space, define open sets and closed sets. Give examples of closed sets, open sets, sets that are both open and closed, and sets that are neither open nor closed.

(3) Show that the intersection of two subspaces of a vector space X is a subspace of X.

(4) Suppose $S := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m\} \subset X$ is a linearly dependent set, where X is a vector space. Show that there exists an element of S that can be expressed as a linear combination of the other elements of S and that span(S) is a subspace of X.

(5) Given a norm $\|\cdot\|$ defined on a vector space X, show that

$$d : X \times X \to \mathbb{R} : (\boldsymbol{x}, \boldsymbol{y}) \mapsto \|\boldsymbol{x} - \boldsymbol{y}\|$$

is a metric.

(6) Let X be a normed space. Show that $(\forall \boldsymbol{x} \in X)(\forall \boldsymbol{y} \in X)\ \|\boldsymbol{x}\| - \|\boldsymbol{y}\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$.

(7) Given an inner product $\langle \cdot, \cdot \rangle$ defined on a vector space X, show that $(\forall \boldsymbol{x} \in X)\ \|\boldsymbol{x}\| := \langle \boldsymbol{x}, \boldsymbol{x} \rangle^{\frac{1}{2}}$ is a norm. (You may assume that that the Cauchy-Schwartz inequality has already been proved.) Is the vector space X equipped with such a norm a Hilbert space?

(8) What is a Cauchy sequence in the context of Hilbert spaces? Do Cauchy sequences in a Hilbert space converge?

(9) In a Hilbert space, define the concepts of weak and strong convergence. When are those concepts equivalent?

(10) In a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, show that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ for every $\boldsymbol{x} \in \mathcal{H}$ implies $\boldsymbol{y} = \boldsymbol{0}$.

(11) Prove the Cauchy-Schwartz inequality.

(12) Let C be a set in a Hilbert space. Show that $C^{\perp}$ is a subspace.

(13) In a Hilbert space, is the intersection of an arbitrary collection of closed sets a closed set? What can we say if the sets under consideration are open?

(14) Define convex sets.

(15) In a Hilbert space, is the intersection of an arbitrary collection of convex sets a convex set?

(16) Suppose that, at time i, a receiver observe the signal:

$$\boldsymbol{r}[i] = b_1[i]\boldsymbol{s}_1 + \sum_{k=2}^{N} b_k[i]\boldsymbol{s}_k + \boldsymbol{n}[i],$$

where k represents an user index, $b_k[i]$ is the transmitted symbol of user k, $\boldsymbol{s}_k \in \mathbb{R}^M$ ($M > N$) is the signature of user k ($\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N$ assumed linearly independent), and $\boldsymbol{n}[i]$ is noise. In the model, $b_k[i]$ and $\boldsymbol{n}[i]$ are samples of random variables/vectors taking values on $\{-1, +1\}$ and $\mathbb{R}^M$, respectively. The random variables corresponding to the transmitted symbols of all users and noise are mutually independent, and all random variables are zero mean. Discuss possible approaches based on linear filters to detect the transmitted bit $b_1[i]$ of the desired user $k = 1$ from $\boldsymbol{r}[i]$.

(17) Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and denote the norm induced by the inner product by $\| \cdot \|$. Solve the optimization problem:

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{x} - \boldsymbol{x}_0\| \\ \text{subject to} & \boldsymbol{x} \in M := \operatorname{span}(\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N\}), \end{array}$$

where $\boldsymbol{x} \in \mathcal{H}$ is the optimization variable, and $\{\boldsymbol{y}_i\}_{i=1,\ldots,N} \subset \mathcal{H}$ and $\boldsymbol{x}_0 \in \mathcal{H}$ are given vectors. Derive the solution when the constraint is replaced by $\boldsymbol{x} \in M^{\perp}$.

(18) Suppose that $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a reproducing kernel Hilbert space (RKHS) with a given kernel $\kappa : E \times E \to \mathbb{R}$ and set E. Solve the following minimization problem (assume that a solution exists):

$$\begin{array}{ll} \text{minimize} & \|f\| \\ \text{subject to} & f(x_1) = y_1 \\ & \quad\vdots \\ & f(x_N) = y_N \end{array}$$

where $\{(x_i, y_i)\}_{i=1,\ldots,N} \subset E \times \mathbb{R}$, $\| \cdot \|$ is the norm of induced by the inner product $\langle \cdot, \cdot \rangle$, and $f \in \mathcal{H}$ is the optimization variable. Is the solution unique if the Gram matrix is not positive definite?

(19) Devise an iterative projection-based method to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$, where $\boldsymbol{A} \in \mathbb{R}^{M \times M}$ and $\boldsymbol{y} \in \mathbb{R}^M$ are given. Assume that, at any given time, you are only able to keep one row of $\boldsymbol{A}$ in the memory of your computer. What can you say about your algorithm when $\boldsymbol{A}$ is singular and nonsingular?

(20) Let $C[-\pi, \pi]$ be the space of real continuous functions defined on $[-\pi, \pi]$ satisfying $\int_{t=-\pi}^{\pi} |x(t)|^2 \, \mathrm{d}t < \infty$. Are the sets S_1, S_2 defined below convex sets?

- $S_1 = \{x \in C[-\pi, \pi] \mid \int_{t=-\pi}^{\pi} x(t)\; y(t)\; \mathrm{d}t = 1\}$, where $y \in C[-\pi, \pi]$ is given.
- $S_2 = \{x \in C[-\pi, \pi] \mid \max_{t \in [-\pi, \pi]} |x(t)| \leq B$, where B is given. (Hint: $\|x\|_\infty := \max_{t \in [-\pi, \pi]} |x(t)|$, $x \in C[-\pi, \pi]$, is a norm.)

(21) In a Hilbert space, does every nonexpansive mapping have a fixed point? Give counter-examples if the answer is negative.

(NOTE: In the following $I : \mathcal{H} \times \mathcal{H} : x \mapsto x$ denotes the identity mapping in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, and $\| \cdot \|$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle$.)

(22) Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a 0.5-averaged nonexpansive mapping. Prove that the mapping $N = 2T - I$ is nonexpansive, and relate the set of fixed points of T with those of N.

(23) Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Prove that the mapping $T : \mathcal{H} \to \mathcal{H}$ given by $T = I + \lambda(N - I)$ is a $\lambda/2$-averaged nonexpansive mapping for every $\lambda \in (0, 2)$ if $N : \mathcal{H} \to \mathcal{H}$ is a 0.5-averaged nonexpansive mapping. Relate the set of fixed points of T with those of N.

(24) Let $T : \mathcal{H} \to \mathcal{H}$ be a quasi-nonexpansive mapping in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. It is known that the set of fixed points of T is given by

$$\mathrm{Fix}(T) := \bigcap_{\boldsymbol{y} \in \mathcal{H}} \left\{ \boldsymbol{x} \in \mathcal{H} \mid \langle \boldsymbol{y} - T(\boldsymbol{y}), \boldsymbol{x} \rangle \leq \frac{\|\boldsymbol{y}\|^2 - \|T(\boldsymbol{y})\|^2}{2} \right\}.$$

Show that $\mathrm{Fix}(T)$ is a closed convex set. You may assume that we already know that "half-spaces" are closed convex sets.

(25) Show that the iteration $x_{n+1} = T(x_n)$ with $x_1 \in \mathcal{H}$ chosen arbitrarily converges weakly to an unspecified point of $\mathrm{Fix}(T)$ (assuming that it is nonempty) if the mapping $T : \mathcal{H} \to \mathcal{H}$ is averaged nonexpansive. (We may assume that the standard Mann iteration has been proven; we only want to show the corollary discussed in the class.)

(26) If $f : \mathcal{H} \to \mathbb{R}$ is a Gâteaux differentiable convex function[1] and f' is Lipschitz continuous [i.e., $(\exists \kappa > 0)(\forall x \in \mathcal{H})(\forall y \in \mathcal{H})\ \|f'(x) - f'(y)\| < \kappa \|x - y\|$].[2] It is known that the mapping $T = I - \mu f'$ for $\mu \in (0, 2/\kappa)$ is averaged nonexpansive. Assume that $\arg\min_{x \in C} f(x) \neq \emptyset$, where $C \neq \emptyset$ is a closed convex set. In such a case, we also know that $I - \mu\ f'$ is an averaged nonexpansive mapping and that $\arg\min_{x \in C} f(x) = \mathrm{Fix}(P_C(I - \mu\ f'))$ for every $\mu > 0$, where $P_C : \mathcal{H} \to C$ is the projection onto C. Show that the sequence $(x_n)_{n \in \mathbb{N}}$ generated by

$$x_{n+1} = P_C(x_n - \mu f'(x_n)), \quad x_1 \in \mathcal{H} \text{ arbitrary},$$

converges weakly to a point $\boldsymbol{x}^\star \in \arg\min_{x \in C} f(x)$ if $\mu \in (0, 2/\kappa)$.

(27) Based on the Mann iteration, prove that the POCS algorithm and the parallel projection method converge weakly if the intersection of the sets is nonempty.

[1] Definition of convexity: $f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2)$ for every $\alpha \in (0, 1)$ and $x_1, x_2 \in \mathcal{H}$

[2] Think of $f'(x)$ as the gradient of f at x (provided that it exists) if you are not familiar with the concept of Gâteaux derivatives

Ex. SS20

(1) metric space $\rightarrow$ has metric

Banach Space $\rightarrow$ metric real vector space ~~with inner product~~
complete normed vector space d induced by $\|\cdot\|$

pre-Hilbert Space Banach + $\langle\cdot,\cdot\rangle$

Hilbert Space pre-Hilbert + complete

Metric space $\supset$ normed real vector space $\supset$ Banach space $\supset$ Hilbert space

norm complete $\langle\cdot,\cdot\rangle$

pre-Hilbert incomplete

(2) open set : $\forall_{x \in S}$ $B(x, \epsilon) := \{y \in E \mid d(x,y) < \epsilon\} \subset S$

$\Rightarrow$ arbitary small distance from point in subset S is still in subset

closed set : $E \setminus S := \{x \in E \mid x \notin S\}$, S open.

$\Rightarrow$ complement of open set

$\Rightarrow$ complete set E and $\emptyset$ is open & closed.

$\Rightarrow$ neither open or closed : $(0, 1]$

(3) vector space X $S_1, S_2 \subset X$

$S_1 \cap S_2 \overset{!}{\subset} X$

$S_1 \cap S_2 \subseteq S_1 \subset X$ $\square$ True by transitivity

(4) $S := \{\vec{x}_1 \dots \vec{x}_n\} \subset X$ lin. dep.

~~$\Rightarrow \sum_{n=1}^{m} \alpha_n \vec{x}_n \neq \vec{0}$~~

~~$\Leftrightarrow \sum_{n=1}^{m} \alpha_n \vec{x}_n = \vec{y}$~~

~~$\Leftrightarrow \sum_{n=2}^{m} \alpha_n \vec{x}_n = \vec{y} - \alpha_1 \vec{x}_1 \quad | \; \vec{y} := (1+\alpha_1)\vec{x}_1 \subset S$~~

~~$\sum_{n=2}^{m} \alpha_n \vec{x}_n = (1+\alpha_1)\vec{x}_1 - \alpha_1 \vec{x}_1 = \vec{x}_1 \not\subset S$ $\square$~~

~~$Span(S) \subset S \subset X$ true by transitivity $\square$~~

Subspace : $x + y \in Span(S)$ true by lin. combination
$\alpha x \in Span(S)$ "

$\forall x, y \in Span(S)$

$\Rightarrow \sum_{k=1}^{m} \alpha_k x_k = 0$, $\exists \alpha_k \neq 0$ (lin. dep)

$\sum_{k=2}^{m} \alpha_k x_k + \alpha_1 x_1 = 0$ | wlog.

$x_1 = \frac{\sum_{k=2}^{m} \alpha_k x_k}{\alpha_1}$ defined, as $\alpha_1 \neq 0$

$\sum_{k=1}^{m} \alpha_k x_k = 0 \Leftrightarrow \alpha_1 \dots \alpha_n = \vec{0}$
lin indep.

(5) X. $\|\cdot\|$. $d: X\times X \mapsto \mathbb{R}: (x,y) \mapsto \|x-y\|$ is metric

metric: $d(x,y)\geq 0$; $d(x,y)=0 \Leftrightarrow x=y$; $d(x,y)=d(y,x)$

$d(x,z) \leq d(x,y) + d(y,z)$

norm: • $\|x\|\geq 0 \Rightarrow \quad z := x-y \quad \Rightarrow \|z\|\geq 0$; $\|x-y\|\geq 0$

• $\|x\|=0 \Leftrightarrow x=0 \quad z:=x-y \quad \|x-y\|=0$

$\Rightarrow \quad x-y=0 \Leftrightarrow x=y$

• $\|x-y\| \overset{!}{=} \|y-x\| \qquad | \|\alpha x\| = |\alpha| \|x\|$, $\alpha=-1$

$\|-x+y\| = |-1| \, \|x-y\| = \|x-y\|$

• $a:=x-y$; $b:=y-z$;

$\|a+b\| \leq \|a\| + \|b\|$

$\|x-\cancel{y}+\cancel{y}-z\| \leq \|x-y\| + \|y-z\|$ $\square$

(6) $\|x+y\| \leq \|x\| + \|y\| \quad |$ Triangle $\quad | x+y=:a$, $b:=y$

$\|a\| \leq \|a-b\| + \|b\|$

$\|a\| - \|b\| \leq \|a-b\|$ $\square$

(7) VS X $(\forall x \in X)$ $\|x\| :\overset{!}{=} \langle x,x\rangle^{\frac{1}{2}}$ shall be a norm

• $\|x\| \overset{!}{\geq} 0 \Leftrightarrow \langle x,x\rangle \geq 0$

• $\langle x,x\rangle = 0 \Leftrightarrow x=0 \Rightarrow \|x\|=0 \Leftrightarrow x=0$

• $\langle \alpha x, \alpha x\rangle = \alpha^2 \langle x,x\rangle = \alpha^2 \|x\|^2 \Rightarrow$

$|\alpha| \|x\| = \sqrt{\alpha^2 \|x\|^2}$

• $\|x+y\| \leq \|x\| + \|y\| \qquad | (\cdot)^2$

$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

$\langle x+y, x+y\rangle \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$

$\langle x,x\rangle + \langle y,y\rangle + 2\langle x,y\rangle \leq$ "

$\|x\|^2 + \|y\|^2 + 2\langle x,y\rangle \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$

$\cancel{2}\langle x,y\rangle \leq \|x\|\|y\|$ by Cauchy-Schwarz

$\square$

(8) Cauchy-Sequence in $\mathcal{H}$

Hilbert space is complete => every Cauchy-sequence converges

(9) $\mathcal{H}$: weak & strong convergence

"$\rightharpoonup$" : $\lim_{n\to\infty} \langle x_n, y\rangle = \langle x^*, y\rangle$

"$\rightarrow$" : $\lim_{n\to\infty} x_n = x^*$

$\mathcal{H}$ is finite-dimensional : "$\rightharpoonup$" $\Leftrightarrow$ "$\rightarrow$"

(10) $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ $(\forall x \in \mathcal{H}) : \langle x, y\rangle = 0 \overset{!}{\Rightarrow} y = \vec{0}$

$z_i :=$ ~~Span~~ Basis $\{\mathcal{H}\}$ $\Rightarrow x = \sum_i \alpha_i z_i \quad \forall x \in \mathcal{H}, \ \alpha_i \in \mathbb{R}$

$\Rightarrow \langle \sum_i z_i, y\rangle = 0$

$\Leftrightarrow \langle z_1, y\rangle + \ldots + \underbrace{\langle z_n, y\rangle}_{I_n} = 0$

x is arbitrary => every Element I_n has to be 0

=> in none ~~Direction~~ Dimension there is a y which would give $\langle y, y\rangle \neq 0$

$\Rightarrow y = 0$

$\square$

(11) Cauchy-Schwarz:

$0 \le \|x - \alpha y\|^2 = \|x\|^2 + \alpha^2\|y\|^2 - 2\alpha\langle x, y\rangle \quad \Big| \ \alpha = \frac{\langle x, y\rangle}{\|y\|^2}$

$= \|x\|^2 + \cancel{\frac{\langle x, y\rangle^2}{\|y\|^2}} - \cancel{2}\frac{\langle x, y\rangle^2}{\|y\|^2}$

$\frac{\langle x, y\rangle^2}{\|y\|^2} \le \|x\|^2$

$|\langle x, y\rangle| \le \|x\|\,\|y\| \quad \square$

(12) $C \subset \mathcal{H}$ (*)

$C^\perp = \bigcap_{y \in C} C(y)$ → arbitrary intersection of closed set is closed set

$C(y) := \{x \in \mathcal{H} \mid \langle x, y\rangle = 0\}$ lin. variety

$C \cap C^\perp = \emptyset$

=> $Base(C) \cap Base(C^\perp) = \emptyset$

=> lin. combinations of $C^\perp$ cannot form a vector of C

(13) $C := \bigcap_i C_i$ $\quad C_i$ closed => C closed

C_i open & finite elements C_i => C open

(14) $(\forall x, y \in S)\ (\alpha x + (1-\alpha) y \in S)$

(15) $C := \bigcap_i C_i$ $\quad C_i$ convex => C convex

(*) $C^\perp = \bigcap_{y \in C} \{x \in \mathcal{H} \mid \langle x, y\rangle = 0\} =: S_i$

intersection of subspaces is subspace

=> S_i Subspace: $y \in C, x \in \mathcal{H}$ $\langle x, y\rangle = 0$

$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle = 0$ ✓

set X which gives $\langle x, y\rangle = 0$

$x \in X \land z \in X \Rightarrow \langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle = 0$

(16)

$$r[i] = b_1[i] s_1 + \sum_{k=2}^{N} b_k[i] s_k + n[i]$$

Matched Filter: $\langle r[i], s_1 \rangle = b_1 + 0 + \langle s_1, n[i] \rangle$

SNR Maximizer: Filter $r = h^T s + h^T n$

$$SNR(h) = \frac{|h^T s|^2}{E(h^T n)^2} = \frac{|h^T s|^2}{h^T R h}$$ $\quad \langle x, y \rangle = x^T R y$ $\quad$ R: Interference matrix

$$= \frac{|\langle h, R^{-1} s \rangle|^2}{\|h\|^2} \overset{CS}{\leq} \frac{\|h\|^2 \|R^{-1} s\|^2}{\|h\|^2} = \|R^{-1} s\|^2 = s^T R^{-1} s$$

$$\Rightarrow h \overset{!}{=} R^{-1} s_1 \cdot \alpha$$ $\quad \Rightarrow \langle s_1, s_k \rangle \approx 0$

Decorrelation Filter $\quad h_c \in \underset{h \in V}{\operatorname{argmin}} \|h\|^2$

$$V = \{ h \in \mathbb{R}^N \mid {\langle h, s_1 \rangle = 1 \atop \langle h, s_k \rangle = 0} \}$$

$\Rightarrow$ noise Amplification

min-Variance Reciever $\quad h \in \underset{h \in H}{\operatorname{argmin}} E(h^T r[i])^2 = \underset{h \in H}{\operatorname{argmin}} h^T R h$

$H := \{ h \in \mathbb{R}^N \mid h^T s = 1 \}$

$\Rightarrow h \in \underset{h \in H}{\operatorname{argmin}} \|h\|^2$ $\quad \Rightarrow H := \{ h \in \mathbb{R}^N \mid \langle h, R^{-1} s \rangle = 1 \}$

$\Rightarrow h = P_H(0)$

(17) $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, $\|\cdot\|$ $\quad \min \|x - x_0\|$ s.t. $x \in M := \operatorname{span}(\{y_1 \dots y_N\})$

$$x^* = P_M(x_0)$$

for $\min \|x - x_0\|$ s.t. $x \in M^\perp$

$$\Rightarrow x^* = P_{M^\perp}(x) = x - P_M(x)$$

$x^* = \sum_{i=1}^{n} a_i y_i$ $\quad$ if $y_{1 \dots N}$ ONB $\Rightarrow \vec{a} = \begin{bmatrix} \langle x, y_1 \rangle \\ \vdots \\ \langle x, y_n \rangle \end{bmatrix}$

(18) $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is RKHS

$$\min \|f\| \text{ s.t. } f(x_1) = y_1, \ \vdots, \ f(x_N) = y_N \quad \Rightarrow \quad \min \|f\| = \|f - 0\| \text{ s.t. } \{\langle f, k(\cdot, x_i) \rangle = y_i\}_{1 \dots N} =: V$$

$\Rightarrow f^* = P_V(0) = P_{M}(x^*)$ $\quad$ ~~$V := \{x \in \mathcal{H} \mid \langle x, y_i \rangle = b_i\}$~~ $\quad M = \operatorname{span}\{y_1 \dots y_N\}$

$= \sum_{i=1}^{N} a_i k(\cdot, x_i)$ $\quad$, a_i by Gram Matrix y_i

Gram-Matrix pos. definite $\Rightarrow$ if only if y_i lin. independent $\Rightarrow$ solution not unique

Define $\mathcal{H}$ / $<\cdot,\cdot>$
$<x,y> := x^T y$
$T := P_1(\dots P_n(x_k))$
$\text{Fix}(T) = \bigcap_k V_k$
$\to$ Mann-Iteration

(19) $Ax = y$ $\quad$ A singular $\to$ no solution / inf. # solutions

$V_k := \{ x \in \mathcal{H} \mid <x, A_i> = y_i \quad \forall i = 1 \dots n \}$

~~$x_{k+1} = P_{V_{k+1}}(x_k)$, $x_1 = P_{V_1}$~~

$x_{k+1} = P_{V_{k+1}}(x_k)$, $\quad x_1 = P_{V_1}(0)$

simultaneous projection method probably faster

(20) $C[-\pi,\pi]$ space of real. cont. functions, power signals

$S_1 = \{ x \in C[-\pi,\pi] \mid \int_{t=-\pi}^{\pi} x(t)\, y(t)\, dt = 1 \}$, $\quad y \in C[-\pi,\pi]$

$\Rightarrow <x,y> := \int_{t=-\pi}^{\pi} x(t)\, y(t)\, dt.$

$\Rightarrow S_1 = \{ x \in C \mid <x,y> = 1 \}$ $\quad y \in C$

$\Rightarrow$ linear Variety $\Rightarrow$ convex

$S_2 = \{ x \in C[-\pi,\pi] \mid \max_{t \in [-\pi,\pi]} |x(t)| \leq B \}$

$\Rightarrow \|\cdot\|_\infty$ is a norm

$\Rightarrow S2 = \{ x \in C \mid \|x\|_\infty \leq B \}$ $\to$ is a Ball
$\to$ convex

(21) $\mathcal{H} = \mathbb{R}^{>1}$ $\quad T: \mathbb{R} \to \mathbb{R} \quad x \mapsto \frac{1}{x}$

$\Rightarrow \|T(x) - T(y)\| \leq \|x - y\|$ $\Rightarrow$ T is nonexpansive

~~T has fixed point~~ $\emptyset$ $\quad \text{Fix}(T) = \emptyset$

This is proof for Projections! $\to$

(22) $(\mathcal{H}, <\cdot,\cdot>)$, $T: \mathcal{H} \to \mathcal{H}$ 0.5-averaged nonexp. mapping

$\Rightarrow T = T_N \cdot \frac{1}{2} + \frac{1}{2} I$, $\quad T_N$ nonexpansive

$N = 2T - I = T_N$ $\Rightarrow$ also nonexpansive, see above

$\text{Fix}(T) = \text{Fix}(T_N) = \text{Fix}(N)$

This is proof for relaxed Projections! $\to$

(23) $T: \mathcal{H} \to \mathcal{H}$, $T = I + \lambda(N - I)$ $\quad \lambda/2$ averaged for $\lambda \in (0,2)$ if N 0.5-averaged non exp.

$N = \frac{1}{2} N_N + \frac{1}{2} I \Rightarrow Q := 2N - I$ $\quad$ nonexpansive $\quad$ (see (22))

$T = I + \lambda(N - I)$

$= I + \frac{\lambda}{2}(2N - 2I)$

$= (1 - \frac{\lambda}{2}) I + \frac{\lambda}{2} \underbrace{(2N - I)}_{Q}$ $\Rightarrow \frac{\lambda}{2}$ averaged nonexp.

$\square$

(24) $T: \mathcal{H} \to \mathcal{H}$ quasi-nonexp.

$$Fix(T) := \bigcap_{y \in \mathcal{H}} \left\{ x \in \mathcal{H} \mid \langle y - T(y), x \rangle \leq \frac{\|y\|^2 - \|T(y)\|^2}{2} \right\}$$

$Fix(T) \stackrel{!}{=}$ convex

$$\frac{\|y\|^2 - \|T(y)\|^2}{2} =: C(y) \quad \text{independant of } x$$

$$y' := y - T(y)$$

$$\Rightarrow Fix(T) = \bigcap_{y \in \mathcal{H}} \{ x \in \mathcal{H} \mid \langle y', x \rangle \leq C(y) \}$$

$\Rightarrow$ intersection of ~~Hyperplanes~~ Halfspace $\Rightarrow$ closed & convex $\square$

(25) $x_{n+1} = T(x_n)$ $\quad x_{n+1} \to x^* \in Fix(T)$

for $T: \mathcal{H} \to \mathcal{H}$ averaged nonexp.

$\Rightarrow T = (1-\alpha) I + \alpha T_N$, T_N nonexpansive

$\Rightarrow x_{n+1} = (1-\alpha) x_n + \alpha T(x_n)$

$\Rightarrow$ converges (Mann-Iteration) $\square$

(26) $f: \mathcal{H} \to \mathbb{R}$, f' Lipschitz cont. with k

① $T_1 = I - \mu f'$, $\mu \in (0, \frac{2}{k})$ averaged nonexpansive

② $\operatorname{argmin}_{x \in C} f(x) = \emptyset$

③ $\operatorname{argmin}_{x \in C} f(x) = Fix(P_C(I - \mu f'))$, $\mu > 0$

from ②: P_C is projection on closed, convex, nonempty set

from ①: T_1, P_C both averaged nonexpansive

$\Rightarrow T' = P_C T_1$ is α' averaged nonexpansive

$\Rightarrow T' = (1-\alpha) I + \alpha T_N'(x)$ Fix point of T' is solution

for $x_{n+1} = (1-\alpha) x_n + \alpha T_N'(x_n) \to x^* \in Fix(T')$

from Mann-Iteration

from ③: $x^* \in \operatorname{argmin}_{x \in C} f(x)$, constraint μ from ①

(27) POCS: $x_{n+1} = P^n(x_n)$, $P^n := P_0 P_1 \dots P_n$

P_i α-averaged nonexp. $\forall i$ $\Rightarrow P^n$ α'-averaged nonexp.

$\Rightarrow x_{n+1} = (1-\alpha) x_n + P^n(x_n) \Rightarrow x^* \in Fix$ (Mann)

SART: $P^n := \sum_{i=1}^{n} \omega_i P_{C_i}$ also averaged nonexp. $\Rightarrow$ converges to Fix