# Possible exam questions for the course
# “Numerische Mathematik 2 für Ingenieure”
# Winter Semester 2014/15 Martin Eigel

Exams will take 20-30 min and will be conducted in English or German. All topics presented in the lecture are relevant, in particular the topics which were covered in the assignments. Main topics are

1. **General theory of PDEs**
2. **Finite Difference Methods**
3. **Finite Element Methods**
4. **Iterative Methods (for linear systems)**

The student may choose to start the oral exam with a presentation of an arbitrary topic of the lecture (no longer than 5 minutes). The presentation may use whiteboard/paper.
In the exam, any topic of the lecture may be discussed. Expect that each topic is touched upon at least with a few questions. Below you will find a list of typical exam questions. Note that these are just examples and the exam will not be restricted to the ones listed.

**General PDE theory**

• What is the general form of a linear PDE of second order?
• What are the three different types of PDEs and how are they determined from the standard form?
• What types of boundary conditions are discussed in the course?
• What is the general form of a linear elliptic PDE?
• What are classical solutions for an elliptic problem with Dirichlet boundary conditions?
• When do we call a PDE well-posed/ill-posed? Can you give examples?
• Can you give examples for exact solutions of elliptic/parabolic equations (1,2 dimension, on the square, on a disc, on the whole space)?
• When can a Fourier series be used as a basis for the analytic solution? How is it done and what are important properties of the basis?
• What is the statement of the discrete maximum principle and what is it used for?

**Finite Difference Method**

• What is the advantage of working with sparse matrices?
• How are finite difference methods derived?
• What are one-sided and central differences?
• How can one construct the global Laplace operator, e.g. with homogeneous Dirichlet conditions, from the difference stencil? What is lexicographical ordering?
• What are properties of the resulting linear algebraic systems; in which cases is the matrix symmetric?
• What are convergence and consistency orders of finite difference methods?
• How do we usually derive consistency orders and what are typical assumptions?
• What does stability mean for a finite difference method?
• When do we have convergence for a finite difference method?
• What is the classical 3-point stencil (1D), 5-point stencil (2D)?
• How can one derive finite difference approximations on nonuniform grids? For what type of problems are nonuniform grids useful?
• How can one implement homogeneous & inhomogeneous Dirichlet boundary conditions?
• How can one implement homogeneous & inhomogeneous Neumann boundary conditions?
• How do we solve the problem of the Poisson problem with homogeneous Neumann conditions not having a unique solution? Under what conditions (compatibility condition) do we have existence of solutions?
• How can one implement periodic boundary conditions?

• What are typical consistency/convergence orders of the Poisson equation?
• How can we discretize parabolic problems in space and time?
• Which advantages and disadvantages do implicit Euler/explicit Euler/Crank-Nicolson have?
• What is their consistency order and what are the assumptions?
• What does the CFL condition state?
• What are advantages and disadvantages of finite difference methods?
• How can we show stability?

**Finite Element Method**
• What is a weak solution? How does it compare to classical solutions? How is it derived?
• When is a weak solution also a classical solution? When is a classical solution also a weak solution?
• What is the variational form of the Poisson equation (linear elliptic) PDE?
• How do we construct the Galerkin approximation? Show that there is an equivalent minimization problem.
• What are the ideas of Galerkin approximation with finite elements and what are fundamental differences to the finite difference method?
• How can one state the variational form in abstract form using bilinear and linear forms?
• What is an admissible decomposition of the domain?
• How are Dirichlet and Neumann boundary conditions treated in the finite element method?
• Why are they called essential and natural boundary conditions?
• What are examples of finite elements (decompositions and functions) treated in the lecture?
• What function spaces are typically used in the statement/analysis of variational forms? Why?
• How are the linear algebraic systems (Galerkin matrix) assembled?
• What is the idea behind shape functions?
• Can you tell/discuss the dimension of the discrete finite element space, provided you are given a triangulation and know that you have piecewise linear/quadratic basis functions? What happens if you have essential boundary conditions on some part of the domain?
• When is a bilinear form/Galerkin matrix symmetric, when is it positive definite?
• What do the lemma of Lax & Milgram and Cea’s Lemma state?
• What is meant by the term “Galerkin orthogonality”?
• State variational forms satisfying the conditions of Lax & Milgram (and not satisfying them).
• State a problem where the bilinear form is nonsymmetric and discuss possible boundary conditions.
• Under which conditions can we prove convergence of finite element methods?
• State the interpolation problem and describe the properties of an interpolation with FEM basis functions. Why is the interpolation error of interest for the analysis of the FEM?
• What are the a priori convergence rates of FEM in different norms? Are these rates always obtained in practice? What are assumptions on data and solution for optimal convergence?
• What can be reasons for a reduced observed convergence? How can it be remedied?
• Describe the principle of adaptive FEM with its different steps.
• What does adaptivity mean and when can it be beneficial? What is an error estimator and which properties are desirable?
• Describe the discussed error estimators.
• In which cases is it useful to consider basis functions which are piecewise polynomials of a higher degree?
• Why does the pure Neumann problem require special considerations? How can one ensure existence and uniqueness of the solution?
• Explain the notion of reference elements.
• How can linear shape functions and their gradients be defined explicitly on the physical elements?

**Iterative methods:**

• What is the energy norm of the error?
• How can we write a variational form as a minimization problem and are these two equivalent?
• How can we write a quadratic minimization problem with linear constraints as a linear equation?
• What is the idea behind the gradient descent (GD) method?
• How fast does the GD method converge? When is its convergence fast, when slow?
• Derive the algorithm, based on the assumption that the direction is the residual.
• What are the ideas behind the CG method? What are Krylov spaces, in which sense is the CG optimal?
• In which norm do we compute/estimate the error for GD/CG and why?
• Describe the main ingredients of the CG method (conjugate direction, Gram-Schmidt conjugation and how they lead to the efficient CG method).
• Let A be an n x n matrix. Why does CG terminate in a finite number of n steps; in which cases can it converge faster?
• How fast does CG converge in general (in terms of the condition number of A)?
• What is the idea behind preconditioned CG and how is it implemented?
• What is the multilevel idea?
• Describe the multigrid method for linear systems. What are possible benefits?

**Typical questions regarding the assignments/practical implementations might be stated as follows:**

**Finite Difference Method**
• How would you implement a Poisson equation with homogeneous Dirichlet boundary conditions in one dimension using finite differences?
• How would you implement Dirichlet/Neumann boundary conditions in the finite difference method (even in higher dimension)?
• What are periodic boundary conditions? What is their interpretation and how would you implement them?
• Assume you have a matrix A corresponding to an elliptic operator L. How can you use it to solve a parabolic problem u' + Lu=0? (u' = derivative with respect to time)
• Discuss various reasons why to consider/not to consider
i) higher order space discretization, ii) higher order time discretization.
Give explicit examples.
• Show the equivalence of Ax=b and min (x,Ax)/2-(b,x).

**Finite Element Method**
• Derive a weak form of the Poisson problem with homogeneous Dirichlet boundary conditions.
• ... with inhomogeneous Neumann boundary conditions on some part of the boundary, homogeneous Dirichlet on some other part of the boundary.
• How do you compute stiffness matrices and mass matrices (1D/2D)?
• How do we account for Dirichlet conditions in the FEM in practice?
• Which computations are necessary to compute the Galerkin matrix for a 2D Poisson problem? Be ready to explain element generation, computation of transformation, computation of local matrices, and construction of the global matrix.
• What is a major difference between the computation of the stiffness matrix in 1D and in 2D? (Linear elements, linear transformation)
• How can we compute the stiffness matrix on a single element using shape functions?
(Linear elements, linear transformation, 1D & 2D)
• How do we treat Neumann conditions and Dirichlet conditions?
• How are piecewise quadratic elements treated in the finite element method? How can we ensure continuity of the basis functions (by construction from shape functions)?

- How many quadratic shape functions do we have in 1D, 2D, 3D on the reference interval, triangle, and tetrahedron?
- What is the role of the variable e2p/n4e (1D, 2D, linear elements, quadratic elements)?
- How do we define basis functions from shape functions (using e2p/4e)?
- How can we treat convection terms, space dependent coefficients (say space dependent diffusion constant)?
- How can convection dominated problems be stabilized?
- Which additional computations are required for the discussed a posteriori error estimators?
- Which elements of the triangulation are selected for refinement and what is required to refine a mesh?
- How does prolongation of solutions between meshes work? When is it possible?
- What are possible approaches to determine the actual error of a computation? Which norms are common?
- How can one determine numerical convergence rates?
- How can we treat parabolic problems using finite elements?
- Are the piecewise linear basis functions (1D) in H1 , H2 , C0 , C1 , C2 on (0,1)?

**Iterative methods**

- Show that the extended system (corresponding to minimization with constraint) is invertible if the constraint has full rank.
- Explain how you can use this constraint to enforce:

i) Dirichlet boundary conditions, ii) the integral over the solution to vanish to get solutions with homogeneous Neumann boundary conditions.

- Derive the gradient descent method
- Derive the CG method
- Derive the PCG method
- Describe the discussed multigrid method