V1)

a) Polstellen bei $z_0 = 0, 2, 4$

Nur $z_0 = 2$ innerhalb Quadrat

$\Rightarrow 2\pi i \frac{2i\, e^{2\pi i}}{2 \cdot (-2)} = \pi$ 5

b) $f(z) = \frac{(z^2+1)^2}{z^2+1} = z^2+1$ analytisch

$\Rightarrow \oint f(z)\,dz = 0$ 5

c) $\int_1^{\frac{1+i}{\sqrt{2}}} \frac{1}{z}\,dz = \ln z \Big|_1^{\frac{1+i}{\sqrt{2}}}$

$= \ln \frac{1+i}{\sqrt{2}} - \ln 1$;

$\ln 1 = 0$

$\ln \frac{1+i}{\sqrt{2}} = \ln e^{i\frac{\pi}{4}} = i\frac{\pi}{4}$

$\Rightarrow \int f(z)\,dz = i\frac{\pi}{4}$ 5

V3)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad 1$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad 1$$

Damit

- $$\int \sin\frac{1}{z}\, dz = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int \frac{dz}{z^{2k+1}} \qquad 1$$

$$\int \cos\frac{1}{z}\, d\cancel{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int \frac{dz}{z^{2k}} \qquad 1$$

Es ist

- $$\int_{|z|=1} \frac{dz}{z^n} = \begin{cases} 2\pi i & \text{für } n=1 \\ 0 & \text{sonst} \end{cases} \qquad 4$$

Damit

$$\int \sin\frac{1}{z}\, dz = 2\pi i \qquad 1$$

$$\int \cos\frac{1}{z}\, dz = 0 \qquad 1$$

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R1) $x^4+10x^2+9=0$

$\Rightarrow (x^2)_{1,2} = -5 \pm 4 = \begin{matrix}-9\\-1\end{matrix}$

$\Rightarrow x_{1,2} = \pm 3i \,, \quad x_{3,4} = \pm i$

$\Rightarrow \int \frac{x^2}{(x+3i)(x-3i)(x+i)(x-i)} \, dx = \lim_{R\to\infty} \int$ **2**

$= 2\pi i \{ \mathrm{Res}(f,i) + \mathrm{Res}\, f(3i) \}$; **4**

$\Rightarrow \mathrm{Res}(f,i) = \frac{-1}{4i(-2i)2i} = -\frac{1}{16i}$ **3**

$\mathrm{Res}(f,3i) = \frac{-9}{6i\cdot 4i\cdot 2i} = \frac{9}{48i}$ **3**

$\Rightarrow \int_{-\infty}^{+\infty} \dots = 2\pi i \left(\frac{9}{48i} - \frac{3}{48i} \right) = \frac{12\pi}{48} = \frac{\pi}{4}$ **2**

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R2) a)

a) $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ & \lambda^2 \end{pmatrix}$ 1

$\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}^3 = \begin{pmatrix} \lambda^2 & 2\lambda \\ & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ & \lambda^3 \end{pmatrix}$ 1

$\Rightarrow$ Vermutung: $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ & \lambda^n \end{pmatrix}$ 2

Bew. mit Ind.:

$\underline{n=1}$: ✓

$\underline{n \to n+1}$: $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}^{n+1} = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}^n \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$

$\underset{\text{Ind. Vor.}}{=}$ $\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ & \lambda^n \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$

$= \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ & \lambda^{n+1} \end{pmatrix}$ ✓ 2

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R2) b)

$$e^{t\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}^k \quad 1$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix}\lambda^k & k\lambda^{k-1}\\ 0 & \lambda^k\end{pmatrix} \quad 1$$

$$= \begin{pmatrix}\sum \frac{(t\lambda)^k}{k!} & \sum \frac{t^k k\lambda^{k-1}}{k!}\\ 0 & \sum \frac{(t\lambda)^k}{k!}\end{pmatrix} \quad 1 \quad = \begin{pmatrix}e^{t\lambda} & *\\ 0 & e^{t\lambda}\end{pmatrix} \quad 1$$

mit

$$* = \sum_{k=1}^{\infty} t \frac{(t\lambda)^{k-1}}{(k-1)!} = t \sum_{j=0}^{\infty} \frac{(t\lambda)^j}{j!} = t e^{\lambda t} \quad 2$$

$$\Rightarrow e^{t\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}} = \begin{pmatrix}e^{t\lambda} & te^{t\lambda}\\ 0 & e^{t\lambda}\end{pmatrix}$$

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$$= e^{t\lambda} \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$$

a) + b) $\Rightarrow$ 12

R3)

System $\Leftrightarrow$ $\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ **2**

EV+EV's:

$\det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 \overset{!}{=} 0$

$\lambda_{1,2} = 1 \pm \sqrt{1-2} = 1 \pm i$ **2**

EVen: $\lambda_1 = 1+i \Rightarrow -i \quad -1 \mid 0$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ **2**

$\lambda_2 = 1-i \Rightarrow i \quad -1 \mid 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ **2**

$\Rightarrow$ allg. komplexe Lösung:

$\vec{x}(t) = c_1 e^{(1+i)t} \vec{v}_1 + c_2 e^{(1-i)t} \vec{v}_2$ **3**

$\Rightarrow$ allg. reelle Lösung:

$\vec{x}(t) = c_1 \operatorname{Re}\left\{ e^t e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} + c_2 \operatorname{Im}\left\{ e^t e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$

$= c_1 e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$ **3**

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