Probmeth

1) sum of i.i.d. Bernoulli RV s.t.

$\forall i \in \{1 \dots n\}$, $X_i \sim Bern(p)$, $X := \sum_{i=1}^{n} X_i$

ⓐ $Pr[X \geq np + \lambda] \overset{(\lambda>0)}{\leq} e^{-nD(p + \frac{\lambda}{n} \| p)}$

proove

ⓑ $Pr[X \leq np - \lambda] \leq e^{-nD(1-p + \frac{\lambda}{n} \| 1-p)}$

$Y_i = 1 - X_i \quad \forall i \in \{1 \dots n\} \quad \sim Bernoulli(1-p)$

$Y = \sum_{i=1}^{n} 1 - X_i = n - \cancel{X} \iff X = n - Y$

$Pr[X \leq np - \lambda] = Pr[n - Y \leq np - \lambda] = Pr[Y \geq -np + \lambda + n]$

$= Pr[Y \geq n(1-p) + \lambda] \underset{ⓐ}{\leq} e^{-nD(1-p + \frac{\lambda}{n} \| 1-p)}$

2) $\forall i \in \{1 \dots n\}$, $X_i \sim Bern(p_i)$, $X := \sum_{i=1}^{n} X_i$,

$E[X] = \sum_{i=1}^{n} p_i =: n\bar{p}$. Proove, that ⓐ can be used by replacing p with $\bar{p}$

Chernoff.

$Pr[X \geq \mu + \lambda] \overset{t>0}{=} Pr[e^{tX} \geq e^{t(\mu+\lambda)}] \leq E[e^{tX}] e^{-t(\mu+\lambda)}$

$\overset{(ind)}{=} \prod_{i=1}^{n} E[e^{tX_i}] e^{-t(\mu+\lambda)}$

$= [\prod_{i=1}^{n} (1-p_i) e^{t \cdot 0} + p_i e^{t \cdot 1})] \cancel{e^{-t(\mu+\lambda)}} \overset{!}{\leq} (1 - \bar{p} + \bar{p} e^t)^n \cdot \cancel{e^{-t(\mu+\lambda)}}$

$\prod_{i=1}^{n} (1 - p_i + p_i e^t) \overset{!}{\leq} (1 - \bar{p} + \bar{p} e^t)^n \quad | \log(\cdot)$

$\log\left(\prod_{i=1}^{n} (1 - p_i + p_i e^t)\right) \leq \log(1 - \bar{p} + \bar{p} e^t) \cdot n$

$\Rightarrow \frac{1}{n} \sum_{i=1}^{n} \log(1 - p_i + p_i e^t) \leq \log(1 - \bar{p} + \bar{p} e^t)$

log $\Rightarrow$ concave $\Rightarrow$ Jensens Inequality: $g\left(\frac{\sum_{i=1}^{n} p_i}{n}\right) \geq \frac{1}{n} \sum_{i=1}^{n} g(p_i)$

□

3) Interference Model:

$$y(t) = a(t) + \sum_{i=1}^{n} h_i b_i(t)$$

$, a, b_i$ BPSK
$h_i \leq 1$, known
decoder: $y(t) \gtrless 0$

$Pr[\text{error}] \leq ?$

$$Pr[\text{error}] = P(a=1)\, Pr[1 + \sum_{i=1}^{n} h_i b_i \leq 0] + Pr(a=-1)\, Pr[-1 + \sum_{i=1}^{n} h_i b_i \geq 0]$$

a uniformly distr. $\to$ $$= \frac{1}{2}\left(Pr\left[\sum_{i=1}^{n} h_i b_i \leq -1\right] + Pr\left[\sum_{i=1}^{n} h_i b_i \geq 1\right]\right)$$

$X := \sum_{i=1}^{n} \underbrace{h_i b_i}_{X_i \in \{\pm h_i\}}$ $\qquad E[X] = E[\sum_{i=1}^{n} h_i b_i] = \sum_{i=1}^{n} h_i \underbrace{E(b_i)}_{0} = 0$ BPSK, uniform

$$\Rightarrow Pr(\text{error}) = \frac{1}{2}\Big(\underbrace{Pr[X \leq -1]}_{b} + \underbrace{Pr[X \geq 1]}_{a}\Big)$$

a : $$Pr[X \geq 1] = Pr[X \geq \underset{\mu}{0} + 1] \leq \exp\left(\frac{-2 \cdot 1^2}{\sum_{i=1}^{n} (h_i \cdot 2)^2}\right) = \exp\left(\frac{-1}{2 \sum_{i=1}^{n} h_i^2}\right)$$

b : $$Y := -X \Rightarrow Pr[X \leq -1] = Pr[Y \geq 1] = Pr[Y \geq \underset{\mu_Z}{0} + 1] \leq \exp\left(\frac{-1}{2\sum_{i=1}^{n} h_i^2}\right)$$ also BPSK, uniform

$$Pr(\text{error}) \leq \frac{1}{2}(a + b) \leq \exp\left(\frac{-1}{2\sum_{i=1}^{n} h_i^2}\right) \underset{|h_i| \leq 1}{\leq} \exp\left(\frac{-1}{2n}\right)$$

5) Fixed point of permutation $\Pi : [1,n] \to [1,n]$.

Var[# Fixed points]?

indicator: we have a fixed point: $X_i = \mathbb{1}\{\Pi(x) = x\}$

number of fixed points: $X = \sum_{i=1}^{n} X_i$

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i)$$

Π chosen uniformly at random. in $[1,n] \to [1,n]$ only one mapping is fixed.

$$\Rightarrow \mathbb{E}(X_i) = \frac{1}{n} \qquad \Rightarrow \mathbb{E}(X) = n \cdot \mathbb{E}(X_i) = 1$$

$$\mathbb{E}(X^2) = \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} \underbrace{\mathbb{E}(X_i X_j)}_{= \underbrace{Pr(X_i = 1)}_{\frac{1}{n}}\, \underbrace{\mathbb{E}(X_i X_j \mid X_i = 1)}_{\frac{1}{n-1}} + Pr(X_i = 0) \cdot 0}$$

$\mathbb{E}(X_i^2) = \mathbb{E}(X_i) = \frac{1}{n}$ // $X_i \in \{0,1\}$

$$= n \cdot \frac{1}{n} + \frac{n \cdot (n-1)}{n(n-1)} = 2$$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = 2 - 1 = 1$

7) $F_X(x) = \Pr[X \leq x]$ CDF. proove: $\mathbb{E}(X) = \int_0^\infty (1 - F_X(x))\,dx$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\, p_X(x)\, dx = \int_0^\infty \underbrace{x}_{u}\, \underbrace{p_X(x)}_{v'}\, dx \quad \text{| nonnegative}$$

$$= x\, P_X(x) \Big|_0^\infty - \int_0^\infty 1 \cdot P_X(x)\, dx \quad \text{| Integration by parts}$$

$$= x \Big|_0^\infty \left(P_X(\infty) - P_X(0) \right) - \int_0^\infty P_X(x)\, dx$$

$$= x \Big|_0^\infty (1 - 0) - \int_0^\infty P_X(x)\, dx \quad \text{| CDF properties}$$

$$= \int_0^\infty 1\, dx - \int_0^\infty P_X(x)\, dx \quad \text{| } \int 1 = x$$

$$= \int_0^\infty \left(1 - P_X(x)\right) dx \quad \square$$

9) b_0, w_0 known. W_k white balls, b_k black balls.

time k: Pick a ball. if $w \rightarrow$ put another w ball in

Proove: $z_k = \frac{W_k}{W_k + B_k}$ is a martingale.

Number of balls at time k: $W_k = z_k (w_0 + b_0 + k) =: z_k \alpha$

$B_k = (1 - z_k)\, \alpha$

total number of Balls: $\alpha := (w_0 + b_0 + k)$

$$\mathbb{E}(z_k \mid z^{k-1}) \xrightarrow{\text{index shift}} \mathbb{E}(z_{k+1} \mid z^k)$$

$$= \mathbb{E}(z_{k+1} \mid z_k = z_k) \quad \text{| information of } z^{k-1} \text{ contained in } z_k, \alpha$$

$$W_{k+1} = \begin{cases} z_k(\alpha - 1) + 1 & \text{, w.p. } z_k \\ z_k(\alpha - 1) & \text{, w.p. } 1 - z_k \end{cases}$$

$$\Rightarrow z_{k+1} = \begin{cases} \frac{z_k(\alpha - 1) + 1}{\alpha} & \text{, w.p. } z_k \\ \frac{z_k(\alpha - 1)}{\alpha} & \text{, w.p. } (1 - z_k) \end{cases}$$

$$\mathbb{E}(z_{k+1} \mid z_k = z_k) = z_k \frac{z_k(\alpha - 1) + 1}{\alpha} + (1 - z_k) \frac{z_k(\alpha - 1)}{\alpha}$$

$$= \cancel{z_k} \left(\cancel{\frac{z_k(\alpha - 1)}{\alpha}} - \cancel{\frac{z_k(\alpha - 1)}{\alpha}} \right) + \frac{z_k}{\alpha} + \frac{z_k(\alpha - 1)}{\alpha} = \frac{z_k \alpha}{\alpha} = z_k \quad \forall z_k$$

$$\mathbb{E}(z_{k+1} \mid z_k) = z_k \xrightarrow{\text{index shift}} \mathbb{E}(z_k \mid z_{k-1}) = z_{k-1}$$

$$= \mathbb{E}(z_k \mid z^{k-1}) = z_{k-1}$$

$$\tilde{z}_k = 1 - z_k \quad \forall k \Rightarrow \mathbb{E}(1 - z_k \mid z^{k-1}) = 1 - \mathbb{E}(z_k \mid z^{k-1}) = 1 - z_{k-1} = \tilde{z}_{k-1} \quad \square$$

8) $X \sim \mathcal{N}(0,1) \Rightarrow p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$

$$M_{X^2}(t) = \mathbb{E}(e^{tX^2}) = \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(x^2 \left(t - \frac{1}{2}\right)\right) dx$$

for convergence of the Integral, $t - \frac{1}{2} < 0. \Rightarrow t < \frac{1}{2}$

10) "Balls & Bins" m balls uniformly at random into n bins

$F :=$ # empty Balls after all balls are thrown

$\Pr[\,|F - \mathbb{E}(F)| \geq \lambda\,] \leq ?$

$X_i := \mathbb{1}\{$ith. bin is empty at the end$\}$

$\Rightarrow F = \sum_{i=1}^{n} X_i \qquad \mathbb{E}(F) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i)$

Un-needed

$= n \cdot \Pr[\text{bin } i \text{ is empty}]$

$= n \Pr[\text{not put } m\text{-th ball into } n\text{-th bin}]^m$

$= n \cdot \left(1 - \frac{1}{n}\right)^m =: \mu$

X_i dependant $\rightarrow$ cannot use Azuma-Hoeffding / Chernoff

$\Rightarrow$ change to ball perspective:

$W_i :=$ bin, that i-th ball falls into.

$\Rightarrow F = g(W_1 \ldots W_i \ldots W_m)$

$|g(w_1 \ldots w_i \ldots w_m) - g(w_1 \ldots w_i' \ldots w_m)| \leq 1$ 1-Lipschitz

$\Rightarrow$ change of one target bin can either change F by 0 or 1.

$\Rightarrow$ Azuma-Hoeffdings Lemma:

$$\Pr[\,|F - \mu| \geq \lambda\,] \leq 2 \exp\left(-\frac{2\lambda^2}{m}\right)$$

4) $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ independant. $\lambda < \mu$.
Which one expect to be larger?

Prove: $\Pr[X \geq Y] \leq \exp(-\sqrt{\mu} - \sqrt{\lambda})^2$

$\Rightarrow$ $Z := X - Y$

$$\Pr[X \geq Y] = \Pr[Z \geq 0] \overset{(t>0)}{=} \Pr[e^{tZ} \geq 1] \leq \frac{M_Z(t)}{1} = M_{X-Y}(t)$$

$$M_{X-Y} = \exp((\lambda - \mu)(e^t - 1))$$

We get the result for $t = \ln\Big(\underbrace{1 + \frac{-(\sqrt{\mu}+\sqrt{\lambda})^2}{\underbrace{\lambda - \mu}_{<0}}}_{>0}\Big)$ >0

6) Cantelli: $\Pr(X - \mu \geq \lambda) \overset{!}{=} \frac{\sigma^2}{\sigma^2 + \lambda^2}$

$X \sim \text{Bern}(p) \Rightarrow \mathbb{E}(X) = p$, $\sigma^2 = p(1-p)$

$\Pr(X - \mu \geq \lambda) = \Pr(X \geq \lambda + \mu)$ | $\rightarrow$ set $\lambda = 1-p$

$= \Pr(X \geq 1) = p$

$$\frac{\sigma^2}{\sigma^2 + \lambda^2} = \frac{p(1-p)^2}{p^2(1-p)^2 + (1-p)^2} = \frac{p}{p+1-p} = p$$

$$\Rightarrow \Pr(X - \mu \geq \lambda|_{\lambda = 1-p}) = p = \frac{\sigma^2}{\sigma^2 + \lambda^2}\Big|_{\lambda = 1-p}$$

negative tailband for $Y := -X$

11)

X Y	Prob
$(-1, 0)$	$\frac{2}{7}$
$(0, 0)$	$\frac{1}{7}$
$(0, 1)$	$\frac{1}{7}$
$(1, 0)$	$\frac{1}{7}$
$(1, 1)$	$\frac{1}{7}$
$(2, 1)$	$\frac{1}{7}$

independance: $P(Y|X) = P(Y)$

$\Rightarrow P(Y=1 | X=2) = 1 \neq P(Y=1) = \frac{3}{7}$

$$\mathbb{E}(\mathbb{E}(Y|X)^2) = \sum_{x \in X} P(X=x)\, \mathbb{E}(Y|X=x)^2$$

$$= \sum_{x \in X} \sum_{y \in Y} P(X=x)\big(P(Y=y|X=x) \cdot y\big)^2$$

$$= \sum_{x \in X} P(X=x)\big(P(Y=1|X=x) \cdot y\big)^2$$

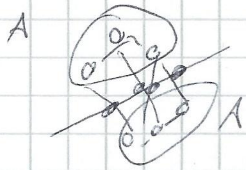
$$= \frac{2}{7} \cdot 0 + \frac{2}{7}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) + \frac{1}{7} \cdot 1^2$$

$$= \frac{2}{7}$$

$\mathbb{E}(\mathbb{E}(X|X^2)^2) \leq \mathbb{E}(\mathbb{E}(Y|X)^2)$ b.c. sign of x is lost.

$$= \frac{2}{7} \cdot \left(\frac{1}{2}\right)^2 + \frac{4}{7} \cdot \left(\left(\frac{1}{4}\right)^2\right) + \frac{1}{7} \cdot (1)^2 = \frac{2}{28} + \frac{1}{28} + \frac{4}{28} = \frac{7}{28} = \frac{1}{4}$$

12) undirected Graph G, n vertices, e edges. $\exists$ cut with value $\geq \frac{e}{2}$



$X :=$ # edges in cut = value of cut (weight = 1 $\forall e$)

$\Rightarrow$ Put each vertex in $A \vee A^c$ uniformly at random.

$$\mathbb{E}(X) = \sum_{i=1}^{e} \mathbb{1}\{\text{edge is in the cut}\}$$

Bernoulli

$$= \sum_{i=1}^{e} \Pr[\text{edge } e_i \text{ is in cut}]$$

$$= \sum_{i=1}^{e} \Pr[v_{e_{i1}} \in A \cap v_{e_{i2}} \in A^c \cup v_{e_{i1}} \in A^c \cap v_{e_{i2}} \in A]$$ (uniform

$$= \sum_{i=1}^{e} \Pr[v_{e_{i1}} \in A \cap v_{e_{i2}} \in A^c] + \Pr[v_{e_{i1}} \in A^c \cap v_{e_{i2}} \in A]$$

$$= \sum_{i=1}^{e} \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$$

$$\mathbb{E}(X) = \frac{e}{2} \Rightarrow \exists \text{ cut with value } \geq \frac{e}{2}$$ e

13) graph G: $N := n - \binom{n}{3} 2^{-3} \cdot 2$ vertices, $n \geq 3$

$\Rightarrow \exists$ a colouring with no mono-chromatic triangles.

$X :=$ # of mono-chromatic triangles out of n vertices.

$$\mathbb{E}(X) = \binom{n}{3} \cdot 2^{-3} \cdot 2$$

combinations of three vertices out of n

all same colour

two available colours.

$\Rightarrow$ From each mono-chromatic triangle, remove one vertex.

$$\mathbb{E}(\underbrace{n - X}_{\text{no monochr. triang.}}) = n - \binom{n}{3} \cdot 2^{-3} \cdot 2 = N$$

$\Rightarrow \exists$ a colouring for N vertices with no mono-chromatic triangles.

14) $X := \sum_{i=1}^{n} X_i \quad , X_i \in \{0,1\}$ can be dependant.

Proove: $\Pr(X \geq 1) \geq \sum_{i=1}^{n} \frac{\Pr(X_i=1)}{\mathbb{E}(X \mid X_i=1)}$

$$y := \begin{cases} \frac{1}{X} & , X > 0 \\ 0 & , \text{else} \end{cases} \qquad \Rightarrow \mathbb{E}(X \cdot y) = \Pr(X > 0) = \Pr(X \geq 1)$$

$$\mathbb{E}(X y) = \mathbb{E}\left(\left(\sum_{i=1}^{n} X_i\right) \cdot y\right) = \sum_{i=1}^{n} \mathbb{E}(X_i y)$$

$$= \sum_{i=1}^{n} \cancelto{0}{\Pr(X_i=0)\, \mathbb{E}(X_i y \mid X_i=0)} + \Pr(X_i=1)\, \mathbb{E}(\underbrace{X_i}_{1} y \mid X_i=1)$$

$$= \sum_{i=1}^{n} \Pr(X_i=1)\, \mathbb{E}\left(\tfrac{1}{X} \mid X_i=1\right)$$

$$\geq \sum_{i=1}^{n} \Pr(X_i=1)\, \frac{1}{\mathbb{E}(X \mid X_i=1)} \qquad \Big|\ \text{Jensen inequ.}\ \tfrac{1}{x} \text{ convex}$$

15) $X_1 \ldots X_n$ independant $\overset{\text{uniform}}{\text{over}}$ $[-1,1]$

$$y := \sum_{i \neq j \in \{1 \ldots n\} : |i-j| \leq 3} X_i X_j \quad *$$

ⓐ $\mathbb{E}(y) = \mathbb{E}\left(\sum X_i X_j\right) = \sum \mathbb{E}(X_i)\mathbb{E}(X_j) = 0$

ⓑ Doop Martingale: $T := f(x_1 \ldots x_n)$

T is 12-Lipschitz: $|f(x_1 \ldots x_i \ldots x_n) - f(x_1 \ldots x_i' \ldots x_n)| \leq \cancel{24}$ $*$

$$\left.\begin{aligned} Z_0 &:= \mathbb{E}(T \mid X_1) \\ Z_1 &:= \mathbb{E}(T \mid X_1, X_2) \\ Z_n &:= f(X_1 \ldots X_n) = y \end{aligned}\right\} \text{Doop - Martingale}$$

$\Rightarrow$ Azuma - Hoeffdings - Lemma

$$\Pr(y > nc) = \Pr(Z_n - \mathbb{E}(Z_n) > nc) = \Pr(Z_n - 0 > nc)$$

$$\leq \exp\left(\frac{-(nc)^2 \cdot 2}{n \cdot 12^2}\right) = \exp\left(\frac{-(\cancel{n}c)^2}{\cancel{n} \cdot 72 \cdot 4}\right)$$