Problem 1: Variational Inference and the ELBO

Consider a generative model with a single continuous latent variable $z \geq 0$ and observed data x. The generative process is defined by an exponential prior distribution over the latent variable:

$$p(z) = \exp(-z) \quad \text{for } z \geq 0$$

Let the likelihood of the data given the latent variable be denoted as $p_\lambda(x|z)$, parameterized by λ.

To approximate the intractable true posterior $p(z|x)$, we introduce an approximate posterior distribution $q_\theta(z|x)$. Assume this variational distribution is a Uniform distribution parameterized by θ (where $\theta > 0$):

$$q_\theta(z|x) = \frac{1}{\theta} \quad \text{for } 0 \leq z \leq \theta$$

Tasks:

1. Formulate the joint distribution of the generative model.
2. Show that the Evidence Lower Bound (ELBO) can be formulated as

$$\mathcal{L}(\theta, \lambda; x) = \mathbb{E}_{q_\theta(z|x)}[\log p_\lambda(x|z)] - D_{KL}(q_\theta(z|x)||p(z))$$

3. Show that

$$D_{KL}(q_\theta(z|x)||p(z)) = -\log\theta + \frac{\theta}{2}$$