Foundations of Stochastic Processes

Oral exam 2023

Task 1: Markov Chains

A tourist moves randomly in a town for 900 hours and switches position every hour. The “map” is shown below. The connection between point 4 and 5 is a bridge.

a) How much time (in average) does he spend in place 9?
b) How often (averagely) does he cross the bridge in direction left-right and how often in direction right-left?

(19) 900 steps

a) $n_{edges}(A) = 2 \qquad \sum_{i=1}^{7} n_{edges\ total}(i) = 9$

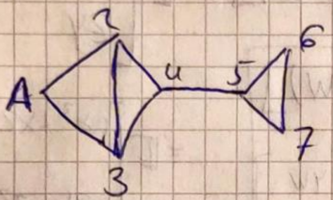
2 poss. to go to A $\qquad$ 2 · 9 poss to go anywhere

$\longrightarrow$ P("going to A") $= \frac{2}{2\cdot 9} = \frac{1}{9}$

$\Longrightarrow E[t_A] = \frac{900}{9} = 100$

&

b)



$P(X_n = i \mid X_{n-1} = j, \dots, X_0 = k)$
$= P(X_n = i \mid X_{n-1} = j)$

$\left(\pi_5 \, p_{54} = \pi_4 \, p_{45} \right)$

$n_{5\to 4} = \pi_5 \cdot p(X_n = 4 \mid X_{n-1} = 5) = \frac{1}{3}\pi_5$

$n_{4\to 5} = \pi_4 \cdot p(X_n = 5 \mid x_{n-1} = 4) = \frac{1}{3}\pi_4$

a) $\pi_A = \frac{n_{edges A}}{2 n_{edges total}} = \frac{1}{9}$

$\pi_4 = \frac{3}{2\cdot 9} = \frac{1}{6}$

$\pi_5 = \frac{3}{2\cdot 9} = \frac{1}{6}$

Task 2: optimal MMSE estimator

X and Z are independent random variables with mean zero and variance \sigma_{x}^2 resp. \sigma_{z}^2. The values Y and W are being observed:

- Y = aX + Z
- W = bX + cZ

(a,b,c constant)

a) Find the optimal estimator for X.
b) Find an expression for the minimum mean square error.

23

$$Y = aX + Z$$
$$W = bX + cZ \quad \rightarrow \quad \overset{\alpha}{\begin{pmatrix} Y \\ W \end{pmatrix}} = \underline{A} \overset{\beta}{\begin{pmatrix} X \\ Z \end{pmatrix}}$$

a) $\hat{x}_{opt} = \hat{x}_{lin}$ (Gaussian)

$$\hat{\beta}_{opt} = \cancel{m_\alpha + \Sigma_{xY}\Sigma_Y^{-1} X} \; m_\beta + \Sigma_{\beta\alpha}\Sigma_\alpha^{-1}(\alpha - m_\alpha)$$

$\cancel{\Sigma_{xY} = E[(}$

$$\Sigma_{\beta\alpha} = E\left[\begin{pmatrix} X \\ Z \end{pmatrix}(Y \; W)\right] = E\begin{bmatrix} XY & XW \\ ZY & ZW \end{bmatrix} = E\begin{bmatrix} aX^2 + XZ & bX^2 + cXZ \\ aZX + Z^2 & bXZ + cZ^2 \end{bmatrix}$$

$$\Sigma_\alpha = E\left[\begin{pmatrix} Y \\ W \end{pmatrix}(Y \; W)\right] = E\begin{bmatrix} aX^2 + 2aXZ + Z^2 & YW \\ WY & bX^2 + 2bcXZ \end{bmatrix}$$

a b
c d

$$\Sigma_\alpha^{-1} = \frac{1}{det(\Sigma)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

b) $E[|\alpha - \hat{\alpha}|^2]$

only for X! $E[|X - \hat{X}|^2]$