R1) $Y(s) := \mathcal{L}[y](s) \Rightarrow$

$$s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = \frac{1}{s^2+1}$$ 3

$$(s^2+4)Y(s) - s = \frac{1}{s^2+1}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}$$ 2

$$= \frac{1}{3}\left\{\frac{1}{s^2+1} - \frac{1}{s^2+4}\right\} + \frac{s}{s^2+4}$$ 2

$$\Rightarrow y(t) = \frac{1}{3}\left\{\sin t - \frac{1}{2}\sin 2t\right\} + \cos 2t$$ 3

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R2) Ansatz: $u(x,t) = f(x)\cdot g(t)$

$$\Rightarrow \quad f''\cdot g = f\cdot g'' \quad \Rightarrow \quad \frac{f''}{f}(x) = \frac{g''}{g}(t)$$

$$\Rightarrow \quad \frac{f''}{f}(x) = \text{const} =: \lambda = \frac{g''}{g}(t)$$

$$f'' - \lambda f = 0 \; 1, \quad g'' - \lambda g = 0 \; 1$$

Sinus-Lösungen nur für $\lambda < 0$, $\lambda =: -k^2$, 1 $k \in \mathbb{R}$

$$f(x) = a_1 e^{ikx} + a_2 e^{-ikx}, \quad g(t) = b_1 e^{ikt} + b_2 e^{-ikt}$$

Einspannbed.: $\Rightarrow \quad f(0) = f(\pi) = 0$

$$\Rightarrow \quad a_1 + a_2 = 0 \quad \Rightarrow \quad a_2 = -a_1$$

$$a_1 e^{ik\pi} + a_2 e^{-ik\pi} = 0 \quad \Rightarrow \quad 2ia_1 \sin k\pi = 0$$

$$\Rightarrow \quad k \in \mathbb{Z} \quad 2$$

$$\Rightarrow \quad u(x,t) = \sum_{k=-\infty}^{+\infty} \sin kx \left(b_{1,k} e^{ikt} + b_{2,k} e^{-ikt} \right)$$

$$= \sum_{k=-\infty}^{+\infty} \sin kx \left(c_k \sin kt + d_k \cos kt \right) \quad 2$$

mit Konstanten c_k, d_k.

zu R2) Anfangsauslenkung $\Rightarrow$

$$u(x,0) = \sum_k \sin kx \cdot d_k \overset{!}{=} \sin 2x + \sin 4x$$

$$\Rightarrow \quad d_2 = d_4 = 1 \,, \quad \text{alle anderen} = 0$$

Anfangsgeschwindigkeit:

$$\frac{du}{dt}(x,0) = \sum_k \sin kx \cdot k \cdot c_k \overset{!}{=} \sin 3x$$

$$\Rightarrow \quad 3 \cdot c_3 = 1 \,, \quad c_3 = \frac{1}{3} \,, \quad \text{alle anderen} = 0$$

Insgesamt:

$$u(x,t) = \sin 2x \cos 2t + \sin 4x \cos 4t + \frac{1}{3} \sin 3x \sin 3t$$

3

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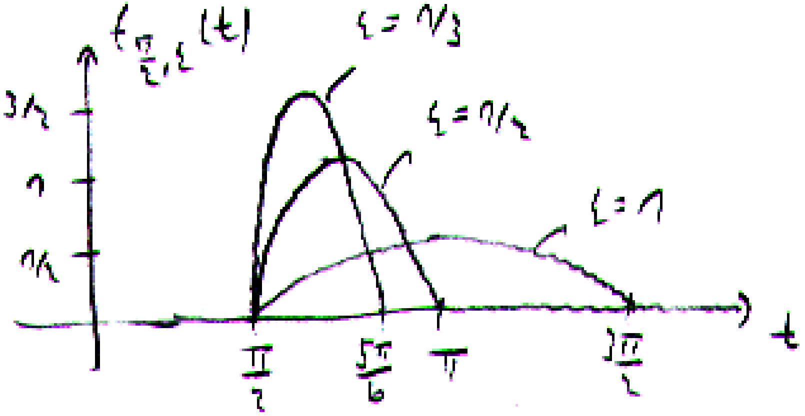
A3) a)



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b) $\hat{f}_{a,\xi}(\omega) = \int_{\mathbb{R}} f_{a,\xi}(t)\, e^{-i\omega t}\, dt$

$= \int_{\mathbb{R}} \frac{1}{\xi} f\left(\frac{t-a}{\xi}\right) e^{-i\omega t}\, dt$ 1 $\quad y = \frac{t-a}{\xi}, \; dy = \frac{dt}{\xi}$

$= \int_{\mathbb{R}} f(y)\, e^{-i\omega(\xi y + a)}\, dy$ 1

$= e^{-i\omega a} \int_0^{\pi} \frac{\sin y}{2} e^{-i\xi\omega y}\, dy$ 1

$= e^{-i\omega a} \cdot \frac{1}{4} \left\{ \frac{e^{-i(\xi\omega-1)y}}{\xi\omega - 1} - \frac{e^{-i(\xi\omega+1)y}}{\xi\omega+1} \right\} \Big|_0^{\pi}$

$= e^{-i\omega a} \frac{1}{4} \left\{ -\frac{e^{-i\xi\omega\pi}}{\xi\omega-1} + \frac{e^{-i\xi\omega\pi}}{\xi\omega+1} - \frac{1}{\xi\omega-1} + \frac{1}{\xi\omega+1} \right\}$

$= e^{-i\omega a} \frac{1}{4} \left(e^{-i\xi\omega\pi} + 1 \right) \frac{-2}{(\xi\omega)^2 - 1}$

R3) b)

$$= \frac{1}{2} e^{-i\omega a} \cdot \frac{1}{1-(\varepsilon\omega)^2} \left(1 + e^{-i\varepsilon\omega\pi}\right)$$ 2

R3) c)

$$\lim_{\varepsilon \to 0} \hat{f}_{\varepsilon,a}(\omega) = e^{-i\omega a}$$ 2

/10

R4) Erzeugende Funktion:

$$\sum_{k=-\infty}^{+\infty} J_k(x)\, t^k = e^{\frac{x}{2}(t-1/t)} \quad 1$$

mit $J_{-k} = (-1)^k J_k \Rightarrow$

$$J_0(x) + \sum_{k=1}^{\infty} \left(t^k + (-1)^k \frac{1}{t^k}\right) J_k(x) = e^{\frac{x}{2}(t-1/t)}$$

$t = i \Rightarrow \quad e^{\frac{x}{2}(i-1/i)} = e^{ix} = \cos x + i \sin x \quad 2$

$$= J_0(x) + \sum_{k=1}^{\infty} \left(i^k + \left(\frac{-1}{i}\right)^k\right) J_k(x) \quad 3$$

$$= J_0(x) + \sum_{k=1}^{\infty} 2 i^k J_k(x)$$

$$= J_0(x) + \sum_{\ell=0}^{\infty} 2 i^{2\ell} J_{2\ell}(x) + \sum_{\ell=0}^{\infty} 2 i^{2\ell+1} J_{2\ell+1}(x) \quad 2$$

$$= J_0(x) + \sum_{\ell=0}^{\infty} 2(-1)^\ell J_{2\ell}(x) + i \sum_{\ell=0}^{\infty} 2(-1)^\ell J_{2\ell+1}(x) \quad 2$$

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V1)

$$\mathcal{L}[t^2 f''(t)](s) = \left(\frac{d}{ds}\right)^2 \mathcal{L}[f''](s) \qquad 1$$

$$= \left(\frac{d}{ds}\right)^2 \{ s^2 F(s) - s f(0) - f'(0) \} \qquad 2$$

$$= 2F(s) + 4s F'(s) + s^2 F''(s) \qquad 1$$

$$\mathcal{L}[f(3t) e^{-2t}](s) = \mathcal{L}[f(3t)](s+2) \qquad 1$$

$$= \frac{1}{3} F\left(\frac{s+2}{3}\right) \qquad 2$$

$$\mathcal{L}\left[\int_0^t \cos(2(t-u)) \cdot f(u)\, du\right](s) = \mathcal{L}[\cos 2t * f(t)](s) \qquad 1$$

$$= \mathcal{L}[\cos 2t](s) \cdot \mathcal{L}[f](s) \qquad 1$$

$$= \frac{s}{s^2+4} \cdot F(s) \qquad 1$$

/10

V2) f
w
f
f
w

V2)

a) falsch 2

b) wahr 2

c) falsch 2

d) falsch 2

e) wahr 2

/10

V3)

$$\sin 2t = -\frac{1}{2}\frac{d}{dt}\cos 2t$$ 2

$$\Rightarrow \mathcal{L}[\sin 2t](s) = -\frac{1}{2}\mathcal{L}[\frac{d}{dt}\cos 2t](s)$$

$$= -\frac{1}{2}\left\{ s \cdot \mathcal{L}[\cos 2t](s) - \cos 0 \right\}$$ 4

$$= -\frac{1}{2}\left\{ \frac{s^2}{s^2+4} - 1 \right\}$$

$$= -\frac{1}{2}\left\{ \frac{-4}{s^2+4} \right\} = \frac{2}{s^2+4}$$ 2

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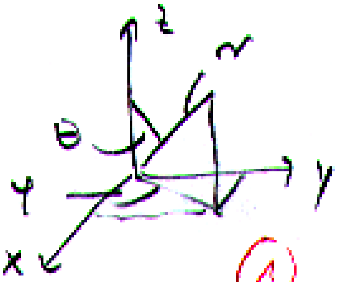
V4) a)



$x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta$ 3

$z = r\cos\theta$

1 2

b) $$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\left\{\frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi} + \cot\theta\,\frac{\Theta'}{\Theta} + \frac{\Theta''}{\Theta}\right\} = \frac{1}{c^2}\frac{\ddot{T}}{T}$$ 2

$$\Rightarrow \frac{1}{c^2}\frac{\ddot{T}}{T} = \lambda_1 \,, \quad \ddot{T} - \lambda_1 c^2 T = 0$$ 1

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} + \{\ldots\} = r^2\lambda_1$$

$$\Rightarrow r^2\frac{R''}{R} + 2r\frac{R'}{R} - r^2\lambda_1 = -\{\ldots\}$$ 2

$$\Rightarrow r^2\frac{R''}{R} + 2r\frac{R'}{R} - r^2\lambda_1 = \lambda_2 \Leftrightarrow r^2R'' + 2rR' - (r^2\lambda_1 + \lambda_2)R = 0$$ 1

und

$$\{\ldots\} = \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi} + \cot\theta\,\frac{\Theta'}{\Theta} + \frac{\Theta''}{\Theta} = -\lambda_2$$

$$\Rightarrow \frac{\Phi''}{\Phi} = -\sin^2\theta\,\lambda_2 - \sin^2\theta\cot\theta\,\frac{\Theta'}{\Theta} - \sin^2\theta\,\frac{\Theta''}{\Theta}$$ 2

$$\Rightarrow \Phi'' - \lambda_3\Phi = 0 \quad \text{und} \quad -\frac{\lambda_3}{\sin^2\theta} = \lambda_2 + \cot\theta\,\frac{\Theta'}{\Theta} + \frac{\Theta''}{\Theta}$$ 0

$$\Rightarrow \Theta'' + \cot\theta\,\Theta' + \left(\lambda_2 + \frac{\lambda_3}{\sin^2\theta}\right)\Theta = 0$$ 1

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