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State Estimation for Robotics. Final Exam

TU Berlin WS 2021/22.
Chair: Robotic Interactive Perception.
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Student name: ________________________

Student's TU Berlin ID: ___________________

"Closed books" (no slides, no cheat-sheet, no source code). No (smart-)phones. No Google search.

Honor code: I hereby certify that I have not given or received aid in the examination: _______

1. (5 pts) Between a rock and a hard place. Mark all that apply:
   ✓ ~~(a)~~ Rotations have three degrees of freedom.
   (b) Rotation matrices have three constraints.
   (c) There exists at least one parametrization of rotations free from constraints and singularities.
   (d) The rotation vector $\boldsymbol{\phi} = \phi\mathbf{a}$ used to define the Lie algebra is free from singularities.

2. (5 pts) The set of all rotations forms:
   (a) a vector space
   (b) a commutative ring
   ✓ ~~(c)~~ a matrix Lie group
   (d) a matrix Lie algebra

3. (5 pts) The set of rigid body motions, $SE(3)$, is called the special Euclidean group of dimension 3 because:
   2
   ~~(a)~~ matrix multiplication is the operation of the group.
   ~~(b)~~ rigid body motions preserve Euclidean distances and angles in 3D space.
   (c) the determinant of the top-left sub-matrix of T has positive determinant.
   (d) all other answers are correct.

4. (5 pts) The Lie algebra associated with $SO(3)$ is given by the set of 3×3 skew-symmetric matrices. If $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^\top$, what is the form of the hat / lift operator? (Hint $\boldsymbol{\phi}^\wedge \mathbf{a} = \boldsymbol{\phi} \times \mathbf{a}$ for all $\mathbf{a} \in \mathbb{R}^3$).

   ✗ (a) $\boldsymbol{\phi}^\wedge = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ I forgot the cross product :(

5. (5 pts) Given two poses, each with associated uncertainty $\{\bar{T}_1, \Sigma_1\}$, $\{\bar{T}_2, \Sigma_2\}$, what are the mean and covariance of the compound pose $\{\bar{T}, \Sigma\}$?

   (a) $\bar{T} = \bar{T}_1 + \bar{T}_2, \ \Sigma = \Sigma_1 + \Sigma_2$

   (b) $\bar{T} = \bar{T}_1\bar{T}_2, \ \Sigma = \Sigma_1 + \Sigma_2$

   ✓ (c) $\bar{T} = \bar{T}_1\bar{T}_2, \ \Sigma = \Sigma_1 + Ad(\bar{T}_1)\Sigma_2(Ad(\bar{T}_1))^\top$

   (d) $\bar{T} = \bar{T}_1\bar{T}_2, \ \Sigma^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1}$

6. (5 pts) The interpolation between two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ in a vector space is given by the convex combination $\mathbf{x}(u) = (1-u)\mathbf{a} + u\mathbf{b}$, with interpolation parameter $u \in [0,1]$. Hence, $\mathbf{x}(0) = \mathbf{a}$, $\mathbf{x}(1) = \mathbf{b}$ and, e.g., $u = 1/2$ produces the midpoint between $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{x}(0.5) = (\mathbf{a}+\mathbf{b})/2$. Given two poses T_a, T_b, which formula(s) is / are correct to interpolate poses between T_a and T_b?

   (a) $T(u) = (1-u)T_a + uT_b$

   × (b) $T(u) = \exp\left(\left(u \ln\left(T_a^{-1}T_b\right)^\vee\right)^\wedge\right) T_a$ $\quad T(u) = T_a^{-1} T_b T_a \neq T_b$

   (c) $T(u) = T_a^{(1-u)} T_b^u$

   → (d) $T(u) = T_a \exp\left(\left(u \ln\left(T_a^{-1}T_b\right)^\vee\right)^\wedge\right)$ $\quad T(1) = T_b$

7. (5 pts) The Baker–Campbell–Hausdorff (BCH) formula... (mark all that apply)

   ✓ (a) Is the solution for Z to the equation $e^X e^Y = e^Z$ for possibly non-commutative X and Y in the Lie algebra of a Lie group.

   2 (b) Is the the name of the formula for interpolation between elements e^X, e^Y in the Lie group given in terms of the X and Y elements in the corresponding Lie algebra.

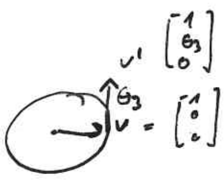
   ✓ (c) Gives an equality of how much $\ln\left(e^X e^Y\right)$ differs from $X+Y$, for X and Y in the Lie algebra of a Lie group.

   × (d) Gives an approximation of how much $\ln\left(e^X e^Y\right)$ differs from $X+Y$, for X and Y in the Lie algebra of a Lie group.

✓ 8. (5 pts) Using the exponential map show that the rotation of the 3D point $\mathbf{v} = (1,0,0)^\top$ by a rotation of angle θ_3 around the Z axis is approximated by the point $\mathbf{v}' = (1, \theta_3, 0)^\top$ on the line $X = 1$. You may draw a diagram to help you reason about the two components of the approximation.

• we know that a full rotation happens with:

$$\exp\left(\begin{bmatrix} 0 \\ 0 \\ \theta_3 \end{bmatrix}^\wedge\right) \cdot v \quad \text{around the } z \text{ axis}$$



• we know that by approximation of the exponential map, we will move along the tangent space of the operating point v:

$$\exp\left(\begin{bmatrix} 0 \\ 0 \\ \theta \end{bmatrix}^\wedge\right) \cdot v \approx \left(I + \begin{bmatrix} 0 \\ 0 \\ \theta \end{bmatrix}^\vee\right) \cdot v = v + v \cdot \begin{bmatrix} 0 \\ 0 \\ \theta \end{bmatrix}^\vee = \begin{bmatrix} 1 \\ \theta_3 \\ 0 \end{bmatrix}$$

9. (10 pts) Letting $\mathbf{C} = \exp(\psi^\wedge)\mathbf{C}_{\text{op}}$, where ψ is a small perturbation applied to an operating "point" (rotation matrix) $\mathbf{C}$, a 3D point $\mathbf{v}$, and $u : \mathbb{R}^3 \to \mathbb{R}$ differentiable, obtain a linear expression for $u(\mathbf{C}\mathbf{v})$ in terms of the vector ψ. Please specify all steps taken.

• first we linearize the inner term of u : $\underbrace{\exp(\varphi^\wedge)}\, Cop$ in which $Cop = \bar{C} \cdot v$

1. we approximate the exponential of the small perturbation by first 2 Taylor terms:

$$\exp(\varphi^\wedge) \cdot Cop \approx (I + \varphi^\wedge) \cdot Cop = Cop + \varphi^\wedge \cdot Cop$$

2. to have a linear term in the perturbation φ, we rearrange the $^\wedge$ operator.

$$= \underbrace{Cop}_{x} + \underbrace{(- Cop^\wedge \cdot \varphi)}_{\delta x}$$ linear in φ → the linearized inner term will be plugged in $u(\cdot)$

• now we linearize the outer term $u(\cdot)$ with help of the first 2 Taylor-terms of $u(x + \delta x)$.

$$u(x + \delta x) \approx u(Cop) + \underbrace{\frac{\partial u}{\partial p}\Big|_{p = Cop}}_{\text{outer Jacobian}} \cdot (\underbrace{- Cop^\wedge}_{\text{inner Jac}} \cdot \varphi)$$ still linear in φ

$$= u(Cop + (- Cop^\wedge \cdot \varphi))$$

10. (10 pts) Suppose that $\mathbf{C} = \exp(\phi^\wedge)\bar{\mathbf{C}}$, with $\phi \sim \mathcal{N}(\mathbf{0}, \mathbf{1})$. Calculate the following:

$$\mu = E\left[\ln\left((\mathbf{C}\bar{\mathbf{C}}^{-1})^2\right)^\vee\right]$$

$$\Sigma = E\left[\left(\ln\left((\mathbf{C}\bar{\mathbf{C}}^{-1})^2\right)^\vee - \overset{0}{\mu}\right)\left(\ln\left((\mathbf{C}\bar{\mathbf{C}}^{-1})^2\right)^\vee - \overset{0}{\mu}\right)^T\right]$$

you may need: $\exp(\alpha \mathbf{z}^\wedge) = \exp(\mathbf{z}^\wedge)^\alpha \in SO(3)$.

$$\mu = E\left[\ln\left((\exp(\phi^\wedge)\underbrace{\bar{C} \cdot \bar{C}^{-1}}_{\substack{I \\ \text{left away}}})^2\right)^\vee\right] = E\left[\ln\left(\exp(\phi^\wedge)^2\right)^\vee\right] = E\left[\ln\left(\exp(2 \cdot \phi^\wedge)\right)^\vee\right]$$

$$= E\left[(2 \cdot \phi^\wedge)^\vee\right] = E[2 \cdot \phi] = 0$$ zero mean

$$\Sigma = E\left[2 \cdot \phi \cdot 2\phi^T\right] = 4 \cdot \underbrace{E[\phi\phi^T]}_{I \,:\, \text{covariance of } \phi} = 4 \cdot I$$

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11. (20 pts) The Extended Kalman Filter (EKF) applied to the estimation of a rotation is described by equations:

$$\check{\mathbf{P}}_k = \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^\top + \mathbf{Q}'_k \tag{1}$$

$$\check{\mathbf{C}}_k = \mathbf{F}_{k-1}\hat{\mathbf{C}}_{k-1} \tag{2}$$

$$\mathbf{K}_k = \check{\mathbf{P}}_k\mathbf{G}_k^\top \left(\mathbf{G}_k\check{\mathbf{P}}_k\mathbf{G}_k^\top + \mathbf{R}'_k\right)^{-1} \tag{3}$$

$$\hat{\mathbf{P}}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{G}_k)\check{\mathbf{P}}_k \tag{4}$$

$$\hat{\mathbf{C}}_k = \exp\left((\mathbf{K}_k(\mathbf{y}_k - \check{\mathbf{y}}_k))^\wedge\right)\check{\mathbf{C}}_k, \tag{5}$$

with $\mathbf{F}_{k-1} = \exp(\Delta t_k \boldsymbol{\omega}_k^\wedge)$, which allow us to compute $(\hat{\mathbf{C}}_k, \hat{\mathbf{P}}_k)$ from $(\hat{\mathbf{C}}_{k-1}, \hat{\mathbf{P}}_{k-1})$ using input angular velocity $\boldsymbol{\omega}_k$ and measurement $\mathbf{y}_k$.

(a) What are $\mathbf{C}, \mathbf{P}, \mathbf{F}, \mathbf{G}, \mathbf{K}, \mathbf{Q}', \mathbf{R}'$?

(b) Please name each equation and describe what it does / represents.

(c) Which equations change with respect to the EKF in vector spaces? Why?

(d) Which operations happen in the Lie group? Which ones in the Lie algebra? Why?

✓ a) · C are the predicted & estimated rotations living in Lie Group $SO(3) \in \mathbb{R}^{3\times3}$
↳ they are the „state" in our application here

· P is the predicted ~~state~~ & estimated state covariance (regarding the rotations) living in the Lie Algebra $so(3) \in \mathbb{R}^{3\times3}$

· F is the state propagation matrix from the motion model and lives in SO(3) Lie Group

· G is the linearized observation model that might map an external point with help of the internal state to the measurement space

· K is the Kalman Gain

· Q' is the noise covariance of the input propagated through the ~~stat~~ prediction model

· R' is the noise of the measurement propagated through the observation model

✓ b) (1) Prediction of the state covariance matrix $\check{P}_k$ by passing previous state cov $\hat{P}_{k-1}$ through the prediction process

(2) State prediction: calculating the predicted mean of our state rotation: $\check{C}_k$ by passing it through the motion model. (it = prev state estimate $\hat{C}_{k-1}$)

(3) calculation of Kalman Gain for current iteration: K_k

(4) Estimation of our state covariance $\hat{P}_k$ by help of $\check{P}_k$, G_k and K_k

(5) Updating the state to obtain optimal estimate $\hat{C}_k$ by retracting the weighted measurement residual into the Lie Group and left multiplying it with our prediction $\check{C}_k$

c) → see whole paper ①

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(11) × c) Change in vector space (live in Lie Algebra : $E[(\delta\varphi)^\wedge(\delta\varphi^\wedge)^T]$

(1) since the covariances and covariance propagation is handled in the Lie Algebra

The equations that change with respect to the vector space EKF are those that involve the <u>mean</u> $\hat{C}_k, \hat{C}_{k-1}$ because now the mean is an element of a Lie Group, not a vectorspace. (2) and (5) change.

(4) because of (1)

(3) because of (1) and (2)

✓ d) <u>which operation is in Lie Group ?</u>

$\underbrace{\exp(\Delta t_k \, \omega_k^\wedge)}_{\text{Rotation}}$

- all of (2) since we perform a group operation $F_{k-1} \hat{C}_{k-1}$
- the outer operation of (5) since its also a Lie Group operation :

$$\underbrace{\exp(\text{vector}^\wedge)}_{\text{proper rotation matrix}} \cdot \check{C}_k$$

<u>which operation is in Lie Algebra ?</u>

- all ~~mentioned~~ mentioned in (11) c) because of the explanation there

→ back to (11) c)

in (5), the residuals of the measurement y_k and projected prediction $\check{y}_k$ also happen in vector space !

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12. (20 pts) Some Simultaneous Localization and Mapping (SLAM) approaches tackle the problem in a "parallel tracking and mapping" (PTAM) fashion, that is, by using two workers (i.e., threads). One worker is responsible for building a map of the environment while the other one is responsible for localizing the sensor (e.g., a camera). Each worker uses the output of the other one as input, in addition to the input data (i.e., video). Let us consider the case of the "localization" worker, also called "tracker" (because it tracks the motion of the sensor with respect to the map). These workers often pursue the goal of finding the pose T that minimizes an objective function $J(T)$. Let

$$J(T) = \frac{1}{2}\|\mathbf{e}(T)\|^2_{\mathbf{W}} \equiv \frac{1}{2}\mathbf{e}^\top \mathbf{W}^{-1}\mathbf{e}$$

where $\mathbf{e}$ represents and error / residual vector (such as pixel coordinates, 3D coordinates, photometric error, etc.) and $\mathbf{W}$ a constant weighting matrix.
Describe how the Lie-Theory flavored Gauss-Newton method deals with this minimization problem: all steps taken to find T^* (the optimal T). Please be *specific* (unambiguous) about the steps and notation (exponentials, perturbations, gradients, Hessian, etc.), and about what operations happen in the Lie algebra and which ones in the Lie group. How does the method guarantee that the solution satisfies the constraints of a pose?

local = car frame

~~in a SLAM task~~

projecting local coords in 3D onto 2D image pixels

- we demonstrate this task with a stereo camera as an example
- assume the nonlinear stereo sensor model $g(\cdot)$ with an outer term g and an inner term, transforming global points into local frames $T \cdot p$ mapper
- the actual stereo camera detects landmarks in image space as measurements y_j j: index of landmark
- the error ~~fun~~ term $e_j(T)$ is then the following:

point p_j is included   pose of ~~robot~~ car

$e_j(T) = y_j - g(T)$ in which the pose T is the to be optimized object of interest

- now stacking $e_j(T)$ for every landmark j over all timesteps k and squaring it with an weight factor according to a NLLS cost function we obtain our full cost:

$J(T) = \frac{1}{2} e^T(T)\, W^{-1} \cdot e(T)$ with $e(T)$ having a rowsize of $2 \cdot j \cdot k$

pixelpoints   landmarks   time steps

- to actually perform optimization here we have to linearize our sensor model first to make everything linear w.r.t. the perturbation between poses: $\delta\varepsilon$

  1. linearizing the inner term: The inner term is as mentioned $T_k \cdot p_{j,k}$ global point
     - we break pose T into nominal and small perturbation: $\exp(\delta\varepsilon^\wedge) \cdot \bar{T}_k \cdot p_{j,k}$ → $\exp(\delta\varepsilon^\wedge)$ is small perturb.
     - and linearize it: $\approx (I + \delta\varepsilon^\wedge)\bar{T}p_j = \underbrace{\bar{T}p_j}_{x} + \underbrace{(\bar{T}p_j)^{\odot}\delta\varepsilon_j}_{\delta x}$ → now this is stuff in Algebra

  2. now we linearize our outer sensor model $g(\cdot) = g(\bar{T}p_j + (\bar{T}p_j)^{\odot}\delta\varepsilon_j)$
     - $g(\bar{T}p_j + (\bar{T}p_j)^{\odot}\delta\varepsilon_j) \overset{\text{first order Taylor}}{\approx} g(\bar{T}p_j) + \frac{g(\cdot)}{p}\Big|_{p=\bar{T}p_j} (\bar{T}p_j)^{\odot}\delta\varepsilon_j$

→ continue on white paper (3)

(12)

- through arriving at the linearized observation model

$$y_j \approx \underbrace{g(\bar T p_j)}_{\beta} + \underbrace{\frac{\partial g(p)}{\partial p}\Big|_{p=\bar T p_j} (\bar T p_j)^{\odot}}_{\delta^T} \underset{\text{linear in } \varepsilon}{\varepsilon_j}$$

δ^T can be interpreted as ~~Jaco~~ Jacobian ≙ gradient^T

- we have a term exactly linear in perturbation ε
- plug this back into our ~~squared cost we have~~ ~~cost~~ error we have:

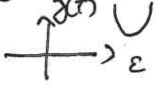
~~$J(T) \approx$~~

$$e_j(T) \approx y_j - (\beta_j + \delta_j^T \cdot \varepsilon_j) = \overbrace{(y_j - \beta_j)}^{z} - \delta_j^T \cdot \varepsilon_j$$

- now insert over all js and timesteps ~~and~~ into our squared cost:

$$J(T) \approx \frac{1}{2}(z - \delta^T \cdot \varepsilon)^T W^{-1} (z - \delta^T \cdot \varepsilon) \quad \leftarrow \text{huge matrix}$$

quadratic in ε →



⇐ parabola

- we see our cost term is quadratic! in ε: approx. J is a parabola in ε!
- we can now do Gauss-Newton: take the derivative and set it to 0 to find the minimum of J at our current operating point $\bar T$

$D(a^T b) = b^T D a + a^T D b$

$$\frac{\partial J(T)}{\partial \varepsilon^T} = W^{-1} \delta (z - \delta^T \cdot \varepsilon) \overset{!}{=} 0$$

$$\Leftrightarrow W^{-1} \delta \delta^T \cdot \varepsilon = W^{-1} \delta z$$ ×

↑ solve for ε

- we finally got our optimal pose perturbation ε in vectorspace!
- now use it with the exponential map to ~~rotate~~ perturb our poses at our current operating point $\bar T$ (over all timesteps) to get a better set of poses:

$$\bar T_{new} = \exp(\varepsilon^\wedge) \cdot \bar T \quad \leftarrow \text{finally returned back to Lie Groups}$$

- we can repeat this whole process until we reached a minimum for our real cost $J(T)$

Nice. You know quite well the reasoning to tackle the problem.

Only; there is just one pose.

$$\frac{\partial J(\tau)}{\partial \varepsilon^T} = (z - S^T \cdot \varepsilon)\, \omega^{-1} (z - S^T \cdot \varepsilon)^T$$

$$\frac{\partial (x - Fb)^T A (x - Fb)^T}{\partial \cancel{x-Fb}\, b}$$

$$\partial A F^T (x - Fb)$$

$$D(a^T \cdot b) = \underline{D a^T \cdot b + D b^T \cdot a}$$

$\cancel{\mathcal{C}}$

$$\ln\left(\exp(\phi^\wedge)\bar{C}\cdot\bar{C}^{-1})^2\right)^\vee = \left(\ln\left(\exp(\phi^\wedge)\, I\right)^2\right)^\vee$$

mean square

σ^2

$$\cancel{E\left[\exp(\phi^\wedge)\exp(\phi^\wedge)\right]}$$

$$\exp(\phi^\wedge)^2 = \exp(2\phi^\wedge)$$

$$E\left[(2\phi^\wedge)^\vee\right]$$

$$= 2\left[E[\phi]\right] = 0$$

$$\left(\ln\left(E\left(C\bar{C}^{-1}\right)^2\right)\right)^\vee$$

$$\Sigma = E\left[\ln\left(\left(C\bar{C}^{-1}\right)^2\right)^\vee\left(\ln\left(\left(C\bar{C}^{-1}\right)^2\right)^\vee\right)^T\right]$$

$$= E\left[\ln\left(\exp(2\phi^\wedge)\right)^\vee\left(\ln\left(\exp(2\phi^\wedge)\right)\right)^{\vee T}\right.$$

$$= E\left[2\phi\cdot 2\phi^T\right] = 4E\cdot\underbrace{\left[\phi\phi^T\right]}_{\Sigma = I}$$